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Rensselaer Polytechnic Institute

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APPLICATION OF MOMENT DISTRIBUTION
TO PRISMATIC AND NON-PRISMATIC
CIRCULAR ARCHED BENTS

WILLIAM W. BARRON AND
ERNEST R. STACEY

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APPLICATION OF
MOMENT DISTRIBUTION TO PRISMATIC AND NON-PRISMATIC
CIRCULAR ARCHED DAMS

By
William W. Barron
and
Ernest R. Stacey

Submitted to the Faculty of
Rensselaer Polytechnic Institute
in partial fulfillment of the requirements
for the degree of Master of Civil Engineering

Troy, New York

May, 1950

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We wish to express our appreciation to Professor
J. Sterling Kinney for his advice as to the preparation
of this thesis.

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INTRODUCTION

OBJECT

This work is an extension of the investigation which was first begun by Cain¹ into the practicability of applying the principles of moment distribution to the solution of arched bents. Cain determined the fixed end moment, carry over factor, and absolute stiffness of prismatic parabolic arches of 25' span for various rise/span ratios. Hansen and Jones² continued the work by investigating various spans and found that for prismatic parabolic arches, a definite relationship existed between these factors and the span for a given rise/span ratio. Mascenik³ did similar work on non-prismatic parabolic arches.

The object of this thesis was to investigate prismatic and non-prismatic circular arches, to determine the fixed end moment, carry over factor, and absolute stiffness for such arches having a rise/span ratio between 0.08 and 0.40, and to apply the moment distribution method of analysis to bents having either type of circular arch as their top member.

¹Cain, E.G., Application of Moment Distribution to Arched Bents, Thesis, E.P.I., February, 1947.

²Hansen, B.L., and Jones, E.G., Method of Application of Moment Distribution to Solution of Arched Bents, Thesis, E.P.I., May, 1948.

³Mascenik, John, Method of Application of Moment Distribution to the Solution of Arched Bents with a Varying Moment of Inertia, Thesis, E.P.I., May, 1949.

DEFINITIONS OF SYMBOLS

- s - Length of span in feet.
- \widehat{L} - Total length of arch in feet.
- r - Rise in feet - distance from springing line to crown.
- H - Horizontal thrust in kips.
- M_R^f - Fixed end moment in kip feet at right springing.
- M_L^f - Fixed end moment in kip feet at left springing.
- V - Vertical reaction in kips.
- I_x - Cross sectional moment of inertia at any point along the arch.
- I_c - Cross sectional moment of inertia at the crown of the arch.
- e - The angle between tangent to working line of arch, and a line parallel to springing line, equal to
- The angle subtended by that portion of the arch between the crown and any point on the arch.
- ds - Increment of length along the arch.
- ΔH - Total horizontal deflection of a point.
- ΔV - Total vertical deflection of a point.
- θ - Total angular rotation of a point.
- δ_{hh} - Horizontal deflection of a point due to the application of a unit horizontal force.
- δ_{vv} - Vertical deflection of a point due to the application of a unit vertical force.
- δ_{hv} - Horizontal deflection of a point due to the application of a unit vertical force.

- δ_{vh} - Vertical deflection of a point due to the application of a unit horizontal force.
- δ'_{hm} - Horizontal deflection of a point due to the application of a unit moment.
- δ'_{vm} - Vertical deflection of a point due to the application of a unit moment.
- α_{em} - Angular rotation of the member at a point due to the application of a unit moment.
- α'_{ev} - Angular rotation of the member at a point due to the application of a unit vertical force.
- α'_{eh} - Angular rotation of the member at a point due to the application of a unit horizontal force.
- y - The vertical distance from the arch center to the neutral point.
- X_{lp} - The horizontal distance from the vertical reference axis to the point of application of the unit load.
- ϕ_{lp} - The angle subtended by the arc between crown and load point. ($\sin^{-1} X_{lp}/R$)
- R - Radius of arch measured in feet.
- H_0 - Horizontal reaction at the neutral point.
- V_0 - Vertical reaction at the neutral point.
- M_0 - Moment at the neutral point.
- L.P. - Load point - point of application of the load.
- A - The angle subtended by the arc between the crown and the right or left springing point.

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SIGN CONVENTION

The carry-over factor is considered positive when the moment applied at the hinged end produces tension on the top of the arch and the moment induced at the fixed end produces tension on the bottom of the arch.

Positive tabulated and plotted fixed end moments indicate tension on the top of the member at the end considered.

The moments considered in the Moment Distribution Method are all internal moments or in other words, the moment exerted by the member on the joint. If the end of the member tends to rotate the joint in a clockwise direction, the internal moment is considered to be positive.

MEMORANDUM

The following is a summary of the information received from the various sources of the above mentioned subject. It is to be noted that the information is of a confidential nature and should be handled accordingly.

Reference is made to the report of the above mentioned subject.

The information received from the above mentioned source is of a confidential nature and should be handled accordingly.

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Reference is made to the report of the above mentioned subject.

PROCEDURE

PROLOGUE

Introduction

It is a well known fact that indeterminate rigid frames may be solved by the method of moment distribution provided the constants (a) stiffness at each end, (b) fixed-end moments at each end, and (c) carry-over factor at each end, are known for each member making up the frame.

Curves giving the required constants for non-prismatic members having one edge straight have been available to designers for some time; however, to the knowledge of the authors, no such complete information has been made available on prismatic and non-prismatic circular arched members.

In this thesis, the constants required for a solution by moment distribution have been determined for prismatic circular arched members, and for non-prismatic circular arched members having rise/span ratios of 0.04, 0.08, 0.20, 0.30, and 0.40.

The exact formula for each of the constants was derived in terms of the geometrical characteristics of the arch, and the relationship between these constants and the span for the same rise/span ratio was determined. The solution of the derived formulas for an arch having a span of one foot provided data for curves from which the required

constants for circular arched members of any span having rise/span ratios within the range of 0.04 to 0.40 can be determined.

The method of analysis used to derive the formula for fixed end moment is a combination of The Conjugate Structure Method and the Neutral Point Method. The General Method and the theory of Virtual Work were used to derive formulas for carry-over factor, absolute stiffness and the effect of spread.

The Conjugate Structure Method was developed by J. Sterling Kinney, Professor of Structural Engineering, R.P.I., Troy, New York. The principles of the method are as follows:

"(1) The conjugate structure, for a given real structure, is identical to the real structure with regard to the lengths of the members and their relative position.

(2) The conjugate structure is positioned in a horizontal plane.

(3) The load, which acts in a vertical direction on the conjugate structure, is the M/EI diagram of the real structure, that is, the conjugate structure is loaded with the flexural strains of the real structure.

(4) If the flexural strain at a given section of the real structure is such as to cause tension on the outside fibers, then this flexural strain is represented as a downward load on the conjugate structure. If compression exists on the outside fibers of the real structure, the load on the corresponding section of the conjugate structure is up.

(5) The conjugate structure, under the action of the real structure flexural strains as loads and the reactions of the conjugate structure, must satisfy three equilibrium condition equations, specifically:

$$\sum M_x = 0$$

$$\sum M_y = 0$$

$$\sum V = 0$$

(6) The shear at any section of the conjugate structure is the slope of the corresponding section of the real structure.

(7) The internal moment on any section of the conjugate structure is the deflection of the corresponding section of the real structure in a direction perpendicular to the lever arm used to find any particular moment.

(8) The end of the conjugate structure corresponding to the end of the real structure which suffers deflection always has a fixed support.

(9) If the moment at any section of the conjugate structure results in tension on the top fiber, the vertical deflection at the corresponding section of the real structure is down and the horizontal deflection is such as to shorten the horizontal projection of the distance between given points in the real structure.

(10) If a section be passed through any point of the conjugate structure and if the portion of the conjugate structure to the right of the section tends to move up,

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with respect to the part on the left of the section, then the rotation of the corresponding section of the real structure is clockwise."¹

It should be pointed out that in this thesis an elastic area, whose width at any point is $1/EI$ for the corresponding section of the real structure, was built up and loaded with an intensity equal to the Moment at the corresponding section on the cut back structure. The product of the load intensity M and the differential elastic area ds/EI gives the differential elastic load on length ds of the conjugate structure. This method provides for a better pictorial representation of the integration for the non-prismatic arch since we do not have to modify the load intensity by $1/EI$.

FIXED END MOMENT AND THRUST DETERMINATION.

By locating the centroid (called the Neutral Point) of the elastic area, cantilevering the arch out from the right end, and connecting the Neutral Point (N.P.) to the left end by a weightless rigid bracket, we can obtain the deflections δ_{vv} , δ_{hh} , and $\alpha_{\theta m}$ as caused by applying individually a 1 kip vertical force, a 1 kip horizontal force, and a 1 foot kip couple at the N.P. We can also evaluate, by principles of the conjugate structure method, the deflections ΔH , ΔV and θ at the N.P. as caused by a 1 kip vertical load placed at any point on the arch.

From the theories of the Neutral Point Method, the reac-

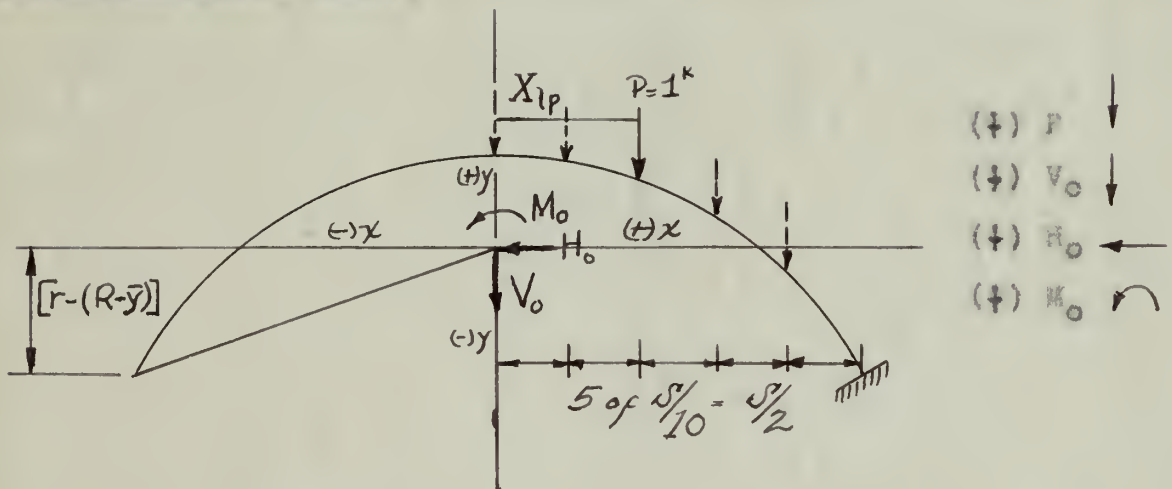
¹ "Indeterminate Structures" - J. Sterling Kinney, August 1949

tions at the K.P. caused by the real kip vertical load on the arch are

$$V_o = (-) \frac{\Delta V}{\delta v v} = V_L \quad H_o = (-) \frac{\Delta H}{\delta h h} = H_L \quad M_o = - \frac{\theta}{\alpha \theta m}$$

Should the value of a reaction come out negative, this indicates that the direction of the reaction is opposite to the direction of the corresponding unit force or couple applied at the K.P.

By applying the reactions at the K.P. and using the sign convention shown below:



$$M_L^f = V_o \left(\frac{S}{2} \right) - H_o [r - (R - \bar{y})] + M_o$$

$$M_R^f = V_o \left(\frac{S}{2} \right) - H_o [r - (R - \bar{y})] + M_o + P \left(\frac{S}{2} - x_{lp} \right)$$

A positive value of moment indicates tension on the top of the arch.

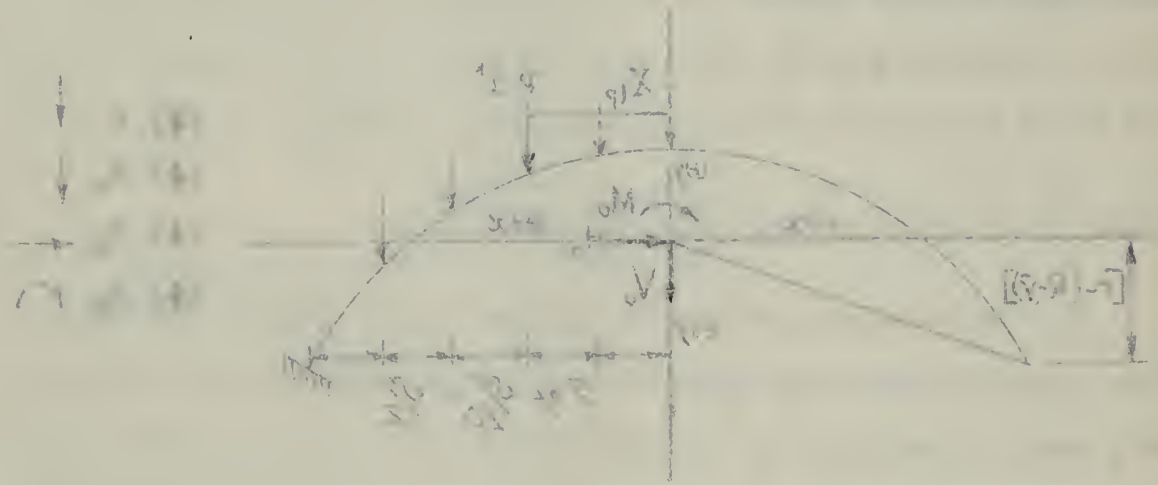
The value of \bar{y} for a prismatic arch = RS/\bar{L} .

The value of \bar{y} for a non-prismatic arch = $\frac{\bar{L}R + S(R-r)}{2S}$

the first value of x for which $y = 0$ and the last value of x for which $y = 0$ are the limits of integration.

$$\frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2} \quad \text{and} \quad \frac{d}{dx} \left(\frac{1}{x^2} \right) = -\frac{2}{x^3}$$

the value of the function at the limits of integration. The value of the function at the upper limit is subtracted from the value of the function at the lower limit. The result is the value of the definite integral.



$$\int_a^b f(x) dx = F(b) - F(a)$$

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

The definite integral of a function $f(x)$ over the interval $[a, b]$ is denoted by $\int_a^b f(x) dx$. The value of the definite integral is the area under the curve $y = f(x)$ from $x = a$ to $x = b$.

CARRY-OVER FACTOR, ABSOLUTE STIFFNESS AND SPREAD

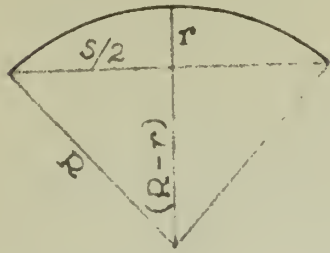
To avoid repetition, the procedure for the determination of these factors is explained on following pages along with the development of the formulas for their values.

Sample calculations for the determination of the fixed end moments, vertical shear, horizontal reactions, carry-over factor, absolute stiffness and reactions due to spread are shown for prismatic arches in Appendix "A", and for non-prismatic arches in Appendix "B".

Tabulated results, with all values expressed in terms of the span length in feet, are shown on pages 56 and 57 for prismatic arches and on pages 65 and 66 for non-prismatic arches.

DEVELOPMENT OF METHOD FOR PRISMATIC ARCH

1. (a) Find the area of the sector of a circle of radius \$r\$ and central angle \$2\alpha\$.



$$r^2 = (r \cos \alpha)^2 + (r \sin \alpha)^2$$

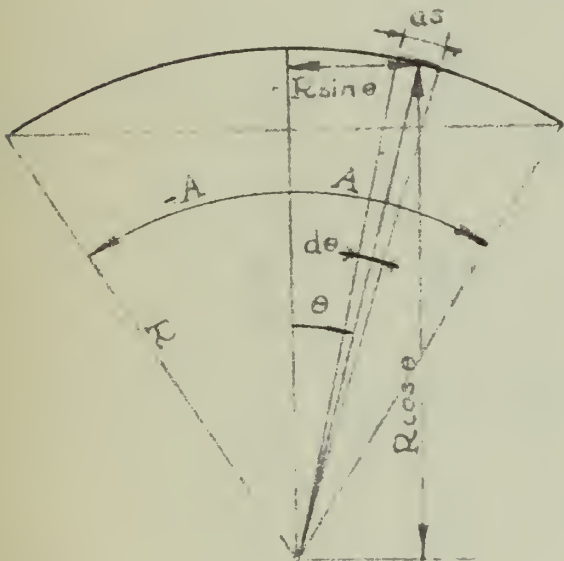
$$r^2 = r^2 \cos^2 \alpha + r^2 \sin^2 \alpha$$

$$r^2 = \frac{r^2 (\cos^2 \alpha + \sin^2 \alpha)}{1}$$

$$r = \frac{r}{1}$$

(b) Find the area of the sector of a circle of radius \$r\$ and central angle \$2\alpha\$.

$$\bar{x} = \frac{\int x \, dA}{\int dA} = \frac{\int_0^\alpha x \, dA}{\int_0^\alpha dA} = \frac{(-) \left[\frac{1}{2} \right]_0^\alpha}{\left[\frac{1}{2} \right]_0^\alpha}$$



$$\bar{x} = \frac{\int x \, dA}{\int dA} = \frac{\int_0^\alpha x \, dA}{\int_0^\alpha dA} = \frac{(-) \left[\frac{1}{2} \right]_0^\alpha}{\left[\frac{1}{2} \right]_0^\alpha}$$

$$= \frac{E \left[\frac{s}{r} - (-) \frac{(-)}{r} \right]}{\frac{s}{r} - (-) \frac{(-)}{r}} = \frac{\left[\frac{s}{r} \right]}{\frac{s}{r}} = \frac{s}{r}$$

2. (a) Find the area of the sector of a circle of radius \$r\$ and central angle \$2\alpha\$.

$$\bar{x} = \frac{\int x \, dA}{\int dA}$$

$$= \frac{\int_0^\alpha x \, dA}{\int_0^\alpha dA}$$

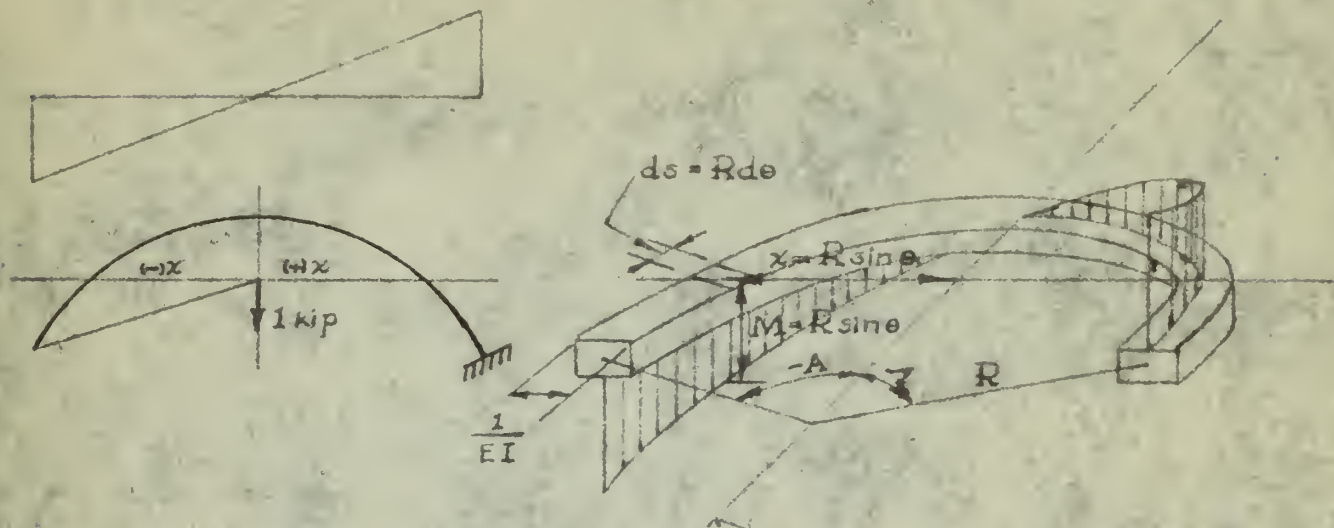
III. DETERMINATION OF DEFLECTION OF NEUTRAL POINT DUE TO:

(A) 1 KIP VERTICAL FORCE AT N.P.

Moment Diagram

Loaded

Conjugate Structure



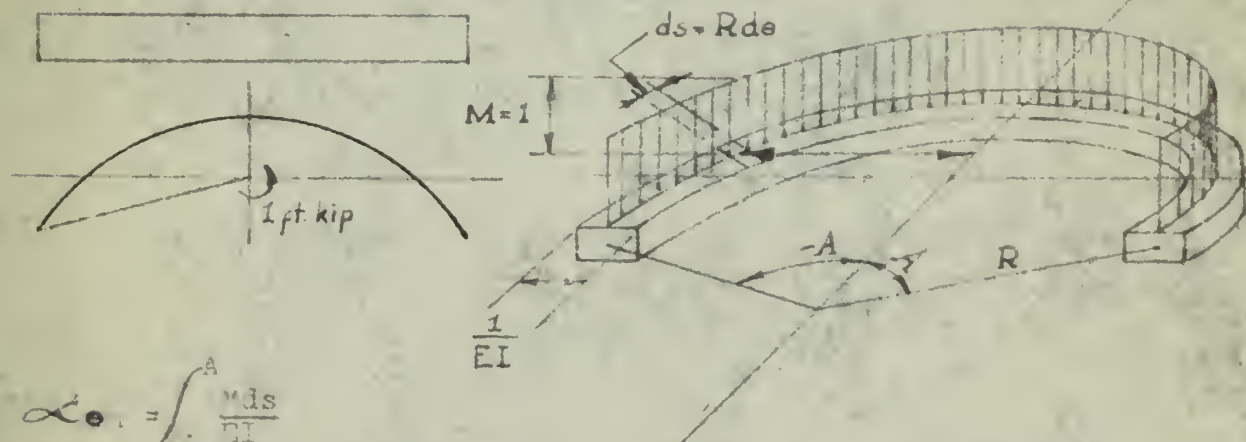
$$\delta_{vv} = \int_{-A}^A \frac{Mds}{EI} (x)$$

$$EI \delta_{vv} = 2 \int_0^A (R \sin \theta) (R d\theta) (R \sin \theta) = 2R^3 \int_0^A \sin^2 \theta d\theta$$

$$= 2R^3 \left[\frac{\theta}{2} - \frac{(\sin \theta)(\cos \theta)}{2} \right]_0^A$$

$$= R^3 \left[\frac{\widehat{L}}{R} - \frac{(R-r)}{R} \right] = \frac{1}{2} [\widehat{L}P - S(R-r)]$$

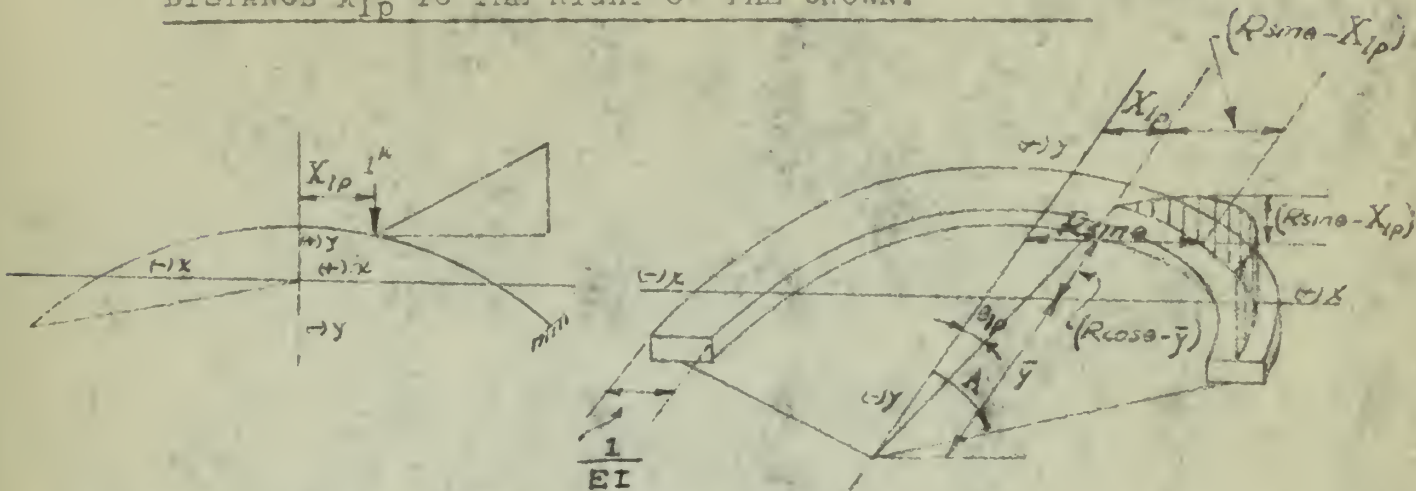
(B) 1 FT. KIP COUPLE AT N.P.



$$\alpha_{em} = \int_{-A}^A \frac{Mds}{EI}$$

$$EI \alpha_{em} = \int_{-A}^A (1) R d\theta = 2 \int_0^A R d\theta = \widehat{L}$$

IV. THE DETERMINATION OF DEFLECTIONS AT THE NEUTRAL POINT (N.P.) CAUSED BY A 1 KIP. VERTICAL LOAD ACTING ON THE ARCH AT A POINT THAT IS HORIZONTAL DISTANCE X_{lp} TO THE RIGHT OF THE CROWN.



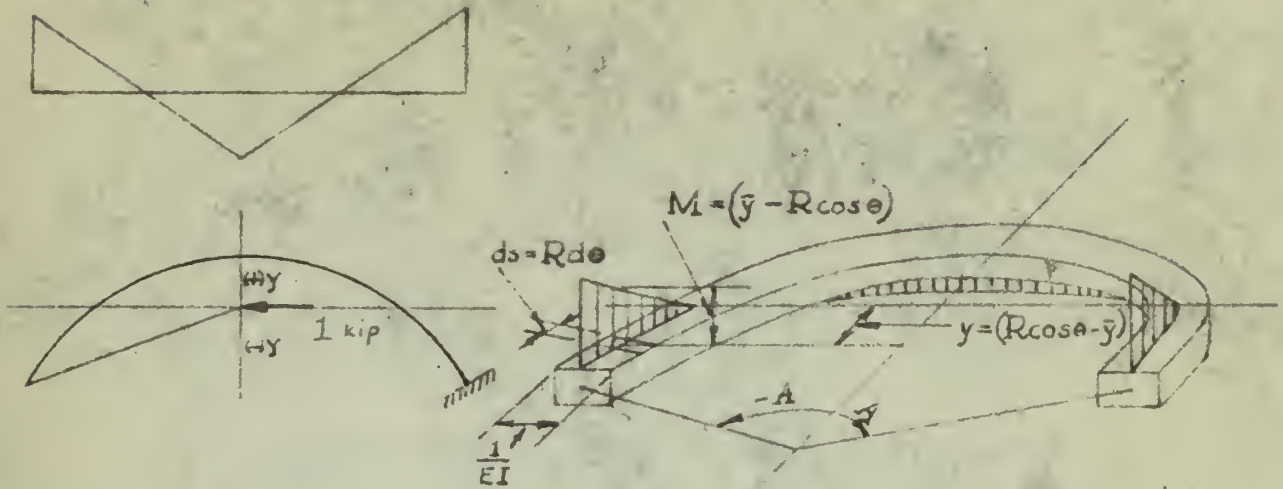
A) DETERMINATION OF VERTICAL DEFLECTION AT N.P.

$$\begin{aligned}\Delta V &= \int_{\theta_{lp}}^A \frac{M ds(x)}{EI} \\ EI \Delta V &= \int_{\theta_{lp}}^A (R \sin \theta - X_{lp}) (R d\theta) (R \sin \theta) \\ &= \int_{\theta_{lp}}^A R^3 (\sin^2 \theta) d\theta - \int_{\theta_{lp}}^A R^2 (X_{lp}) (\sin \theta) d\theta \\ &= R^3 \left[\frac{\theta - (\sin \theta)(\cos \theta)}{2} \right]_{\theta_{lp}}^A + R^2 (X_{lp}) [\cos \theta]_{\theta_{lp}}^A \\ &= \frac{R^3}{2} \left[\frac{\pi}{2} - \frac{S}{2R} \left(\frac{R-r}{R} \right) \right] - \frac{R^3}{2} [\theta_{lp} - (\sin \theta_{lp})(\cos \theta_{lp})] + R^2 (X_{lp}) \left[\frac{R-r}{R} \right] \\ &\quad - R^2 (X_{lp}) (\cos \theta_{lp}) \\ &= \frac{R^3}{4} [\pi - S(R-r)] - \frac{R^3}{2} [\theta_{lp} - (\sin \theta_{lp})(\cos \theta_{lp})] + R(X_{lp})(R-r) \\ &\quad - R^2 (X_{lp}) (\cos \theta_{lp})\end{aligned}$$

(B) DETERMINATION OF HORIZONTAL DEFLECTION AT N.P.

$$\begin{aligned}\Delta H &= \int_{\theta_{lp}}^A \frac{M ds(y)}{EI} \\ EI \Delta H &= \int_{\theta_{lp}}^A (R \sin \theta - X_{lp}) R d\theta (R \cos \theta - \bar{y}) \\ &= \int_{\theta_{lp}}^A R^3 (\sin \theta)(\cos \theta) d\theta - \int_{\theta_{lp}}^A R^2 (\bar{y}) (\sin \theta) d\theta - \int_{\theta_{lp}}^A R^2 (X_{lp}) (\cos \theta) d\theta \\ &\quad + \int_{\theta_{lp}}^A R(\bar{y})(X_{lp}) d\theta\end{aligned}$$

(c) 1 KIP HORIZONTAL FORCES AT N.P.



$$\delta_{hh} = \int_{-A}^A \frac{M^2}{EI} dy$$

$$EI \delta_{hh} = 2 \int_0^A (\bar{y} - R \cos \theta) R d\theta (R \cos \theta - \bar{y})$$

$$= 2 \int_0^A [(-)(\bar{y})^2 + 2(\bar{y})R \cos \theta - R^2 \cos^2 \theta] R d\theta$$

$$= 2 \left[(-)R(\bar{y})^2 \int_0^A d\theta + 2R^2(\bar{y}) \int_0^A \cos \theta d\theta - R^3 \int_0^A \cos^2 \theta d\theta \right]$$

$$= 2 \left[(-)R(\bar{y})^2 \left[\theta \right]_0^A + 2R^2(\bar{y}) \left[\sin \theta \right]_0^A - R^3 \left(\frac{\theta}{2} + \frac{\sin \theta \cos \theta}{2} \right) \right]$$

$$= 2 \left[(-)R \left(\frac{R^2 S^2}{L} \right) \left(\frac{L}{2R} \right) + 2R^2 \left(\frac{LS}{L} \right) \left(\frac{S}{2R} \right) - \frac{R^5}{2} \left[\frac{L}{2R} + \frac{S}{2R} \left(\frac{R-r}{R} \right) \right] \right]$$

$$= (-) \frac{R^2 S^2}{L} + \frac{2R^2 S^2}{L} - \frac{R}{2} [LR + S(R-r)]$$

$$= \frac{R^2 S^2}{L} - \frac{P}{2} [LR + S(R-r)]$$

DETERMINATION OF HORIZONTAL DEFLECTION AT N.P. (Cont'd)

$$\begin{aligned}
 &= R^3 \left[\frac{\sin^2 \theta}{2} \right]_{\theta_{1p}}^A + R^2(\bar{y}) \left[\cos \theta \right]_{\theta_{1p}}^A - R^2(X_{1p}) \left[\sin \theta \right]_{\theta_{1p}}^A + R(\bar{y})(X_{1p}) \left[\theta \right]_{\theta_{1p}}^A \\
 &= \frac{R^3}{2} \left[\frac{S}{2R} \right]^2 - \frac{R^3}{2} (\sin^2 \theta_{1p}) + R^2 \left(\frac{RS}{L} \right) \left(\frac{R-r}{R} \right) - R^2 \left(\frac{RS}{L} \right) \cos \theta_{1p} - R^2(X_{1p}) \left(\frac{S}{2R} \right) \\
 &\quad + R^2(X_{1p}) (\sin \theta_{1p}) + R \left(\frac{RS}{L} \right) (X_{1p}) \left(\frac{\bar{L}}{2R} \right) - R \left(\frac{RS}{L} \right) (X_{1p}) \theta_{1p} \\
 &= \frac{RS^2}{8} - \frac{R^3}{2} (\sin^2 \theta_{1p}) + \frac{R^2 S}{L} (R-r) - \frac{R^3 S}{L} (\cos \theta_{1p}) - \frac{R(X_{1p})S}{2} \\
 &\quad + R^2(X_{1p}) (\sin \theta_{1p}) + \frac{RS}{2} (X_{1p}) - \frac{R^2 S}{L} (X_{1p}) \theta_{1p} \\
 &= \frac{RS^2}{8} - \frac{R^3}{2} \left(\frac{X_{1p}}{R} \right)^2 + \frac{R^2 S}{L} (R-r) - \frac{R^3 S}{L} (\cos \theta_{1p}) + R^2(X_{1p}) \left(\frac{X_{1p}}{R} \right) - \frac{R^2 S}{L} (X_{1p}) \theta_{1p} \\
 &= \frac{RS^2}{8} - \frac{R(X_{1p})^2}{2} + \frac{R^2 S}{L} (R-r) - \frac{R^3 S}{L} (\cos \theta_{1p}) + R(X_{1p})^2 - \frac{R^2 S}{L} (X_{1p}) \theta_{1p} \\
 \Delta H &= \frac{RS^2}{8} + \frac{R(X_{1p})^2}{2} + \frac{R^2 S}{L} (R-r) - \frac{R^3 S}{L} (\cos \theta_{1p}) - \frac{R^2 S}{L} (X_{1p}) \theta_{1p}
 \end{aligned}$$

(C) DETERMINATION OF ANGULAR DEFLECTION AT N.P.

$$\begin{aligned}
 \theta &= \int_{\theta_{1p}}^A \frac{R ds}{EI} \\
 &= \int_{\theta_{1p}}^A (R \sin \theta - X_{1p}) R d\theta \\
 &= \int_{\theta_{1p}}^A R^2 (\sin \theta) d\theta - \int_{\theta_{1p}}^A R(X_{1p}) d\theta \\
 &= (-) R^2 \left[\cos \theta \right]_{\theta_{1p}}^A - R(X_{1p}) \left[\theta \right]_{\theta_{1p}}^A \\
 &= -R^2 \left(\frac{R-r}{R} \right) + R^2 (\cos \theta_{1p}) - R(X_{1p}) \left(\frac{\bar{L}}{2R} \right) + R(X_{1p}) \theta_{1p} \\
 \theta &= -R(R-r) + R^2 (\cos \theta_{1p}) - \frac{(X_{1p})\bar{L}}{2} + R(X_{1p}) \theta_{1p}
 \end{aligned}$$

(D) THE DETERMINATION OF THE REACTIONS AT H.P.

$$V_o = (-) \frac{\Delta V}{\delta_{vv}} \quad , \quad H_o = (-) \frac{\Delta H}{\delta_{hh}} \quad , \quad M_o = - \frac{\Delta M}{\delta_{\theta\theta}}$$

$$V_o = (-) \frac{\frac{R}{4} [\widehat{LR} - S(R-r)] - \frac{R^3}{2} [\theta_{1p} - (\sin \theta_{1p})(\cos \theta_{1p})] + R(\lambda_{1p})(R-r) - R^2(\lambda_{1p})(\cos \theta_{1p})}{\frac{R}{2} [\widehat{LR} - S(R-r)]}$$

$$H_o = (-) \frac{\frac{R^3}{8} + \frac{R(\lambda_{1p})^2}{2} + \frac{R^2 S}{L}(R-r) - \frac{R^3 S}{L}(\cos \theta_{1p}) - \frac{R^2 S}{L}(\lambda_{1p})\theta_{1p}}{\frac{R^2 S^2}{L} - \frac{R}{2} [\widehat{LR} + S(R-r)]}$$

$$M_o = (-) \frac{(-)R(R-r) + R^2(\cos \theta_{1p}) - \frac{(\lambda_{1p})\widehat{L}}{2} + R(\lambda_{1p})\theta_{1p}}{\widehat{L}}$$

(E) THE DETERMINATION OF THE FIRED END MOMENTS

Considering the assumed directions of the reactions at the H.P. on Page 9.

$$M_1^F = (+) V_o \left(\frac{S}{2} \right) - H_o \left[r - (R - \bar{y}) \right] + M_o$$

$$(+) V_o \left(\frac{S}{2} \right) - H_o \left[r - \left(R - \frac{R^2}{L} \right) \right] + M_o$$

M_1^F is determined by writing moments about the right end.

A positive value of moment indicates tension on the top of the arch.

$$\frac{\Delta}{\Delta^2} = \frac{\Delta}{\Delta^2} = \frac{\Delta}{\Delta^2}$$

$$\frac{[\dots]}{[\dots]}$$

$$\frac{[\dots]}{[\dots]}$$

$$\frac{[\dots]}{[\dots]}$$

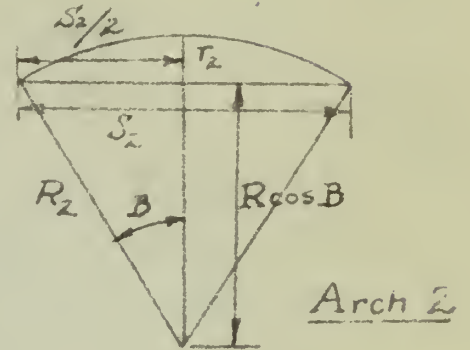
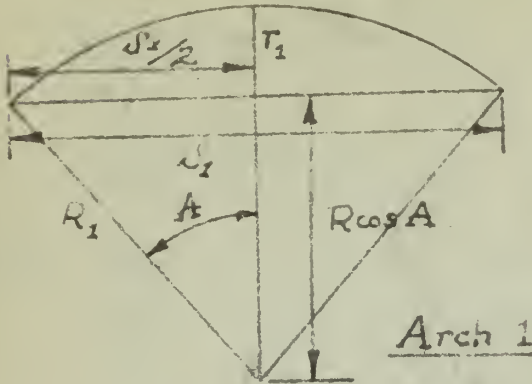
... ..

$$[\dots] + [\dots]$$

... ..

V. DETERMINATION OF RELATIONSHIP BETWEEN MOMENT AND SPAN FOR THE SAME $\frac{\text{RISE}}{\text{SPAN}}$ RATIO.

Proof that arches of same $\frac{\text{Rise}}{\text{Span}}$ ratio are similar.



$$r_1 = R_1 - R_1 \cos A = R_1 (1 - \cos A)$$

$$r_2 = R_2 - R_2 \cos B = R_2 (1 - \cos B)$$

$$\frac{S_1}{2} = R_1 \sin A$$

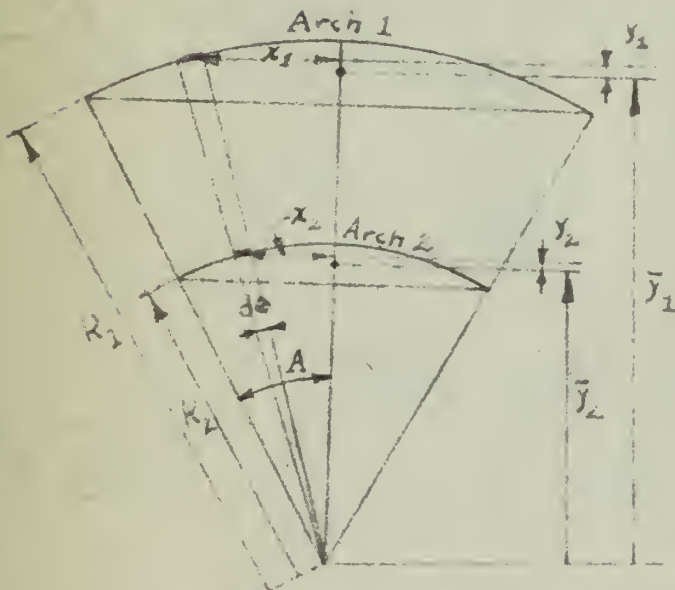
$$\frac{S_2}{2} = R_2 \sin B$$

$$\frac{r_1}{S_1/2} = \frac{1 - \cos A}{\sin A} = \frac{\text{Rise}}{\text{Span}}$$

$$\frac{r_2}{S_2/2} = \frac{1 - \cos B}{\sin B}$$

$$\frac{1 - \cos A}{\sin A} = \frac{1 - \cos B}{\sin B}$$

$\therefore A = B$ AND THE ARCHES ARE SIMILAR.



$$\frac{x_2}{x_1} = \frac{R_2}{R_1} = \frac{S_2}{S_1} \therefore x_2 = x_1 \frac{S_2}{S_1}$$

$$\frac{\bar{y}_2}{\bar{y}_1} = \frac{\frac{R_2 S_2}{R_2 A}}{\frac{R_1 S_1}{R_1 A}} = \frac{S_2}{S_1}$$

$$\frac{y_2 + \bar{y}_2}{y_1 + \bar{y}_1} = \frac{S_2}{S_1}$$

$$\therefore \frac{y_2}{y_1} = \frac{S_2}{S_1} \text{ or } y_2 = y_1 \frac{S_2}{S_1}$$

\therefore The moment arms of corresponding $\frac{M ds}{EI}$ loads vary as $\frac{S_2}{S_1}$

From Page 14 it is seen that M at any point due to a 1^k load on the Arch = $(R \sin \theta - x_{1p})$

$$\therefore \text{For Arch 2, } M_2 = R_2 \sin \theta - R_2 \sin \theta_{1p} = R_2 (\sin \theta - \sin \theta_{1p})$$

$$\text{For Arch 1, } M_1 = R_1 \sin \theta - R_1 \sin \theta_{1p} = R_1 (\sin \theta - \sin \theta_{1p})$$

$$\therefore \frac{M_2}{R_1} = \frac{R_2}{R_1} = \frac{S_2}{S_1} \quad \text{or} \quad M_2 = M_1 \left(\frac{S_2}{S_1} \right)$$

Note: The above M 's are moment intensities acting on areas ds , or the moment existing at the corresponding section of the cut back structure.

$$ds_2 = \frac{R_2}{R_1} ds_1 = \frac{S_2}{S_1} ds_1$$

The Elastic Loads $\frac{Mds}{EI}$ vary as $\left(\frac{S_2}{S_1} \right)^2$

$$\text{Since: } (\Delta V)_1 = \int_{x_{1p}}^A \left(\frac{Mds}{EI} \right)_1 (x_1) \quad \text{and} \quad (\Delta V)_2 = \int_{x_{1p}}^A \left(\frac{Mds}{EI} \right)_2 (x_2)$$

$$\frac{(\Delta V)_2}{(\Delta V)_1} = \frac{\int_{x_{1p}}^A \left(\frac{Mds}{EI} \right)_1 \left(\frac{S_2}{S_1} \right)^2 (x_1) \left(\frac{S_2}{S_1} \right)}{\int_{x_{1p}}^A \left(\frac{Mds}{EI} \right)_1 (x_1)} = \left(\frac{S_2}{S_1} \right)^3$$

By the same reasoning $(\delta_{vv})_2$ as caused by 1 kip vertical

$$\text{at (N.P.)}_2 = (\delta_{vv})_1 \left(\frac{S_2}{S_1} \right)^3$$

$$\text{Hence } \frac{(V_o)_2}{(V_o)_1} = \frac{(-) \frac{(\Delta V)_2}{(\delta_{vv})_2}}{(-) \frac{(\Delta V)_1}{(\delta_{vv})_1}} = \frac{\frac{(\Delta V)_1 \left(\frac{S_2}{S_1} \right)^3}{(\delta_{vv})_1 \left(\frac{S_2}{S_1} \right)^3}}{\frac{(\Delta V)_1}{(\delta_{vv})_1}} = 1 \text{ or } (V_o)_2 = (V_o)_1$$

By similar reasoning $(H_o)_2 = (H_o)_1$

Thus under the action of similarly placed equal loads, the vertical and horizontal reactions at (N.P.)₁ are the same as those at (N.P.)₂.

From (1) it is seen that $\frac{d}{dt} \left(\frac{1}{r} \right) = -\frac{1}{r^2} \frac{dr}{dt}$

$$\frac{d}{dt} \left(\frac{1}{r} \right) = -\frac{1}{r^2} \frac{dr}{dt}$$

$$\therefore \frac{d}{dt} \left(\frac{1}{r} \right) = -\frac{1}{r^2} \frac{dr}{dt} = -\frac{1}{r^2} \left(\frac{dr}{dt} \right)$$

$$\frac{d}{dt} \left(\frac{1}{r} \right) = -\frac{1}{r^2} \frac{dr}{dt} = -\frac{1}{r^2} \left(\frac{dr}{dt} \right)$$

$$\therefore \frac{d}{dt} \left(\frac{1}{r} \right) = -\frac{1}{r^2} \frac{dr}{dt} = -\frac{1}{r^2} \left(\frac{dr}{dt} \right)$$

Now, let us consider the motion of a particle in a circular orbit.

The velocity of the particle is given by $v = r \omega$

$$v = r \omega \quad \therefore \frac{dr}{dt} = r \frac{d\omega}{dt}$$

$$\frac{dr}{dt} = r \frac{d\omega}{dt}$$

$$\frac{dr}{dt} = r \frac{d\omega}{dt} \quad \therefore \frac{dr}{r} = \frac{d\omega}{\omega}$$

$$\frac{dr}{r} = \frac{d\omega}{\omega} \quad \therefore \frac{dr}{r} = \frac{d\omega}{\omega}$$

Integrating both sides, we get $\ln r = \ln \omega + \ln C$

$$\ln r = \ln \omega + \ln C$$

$$\ln r = \ln \omega + \ln C \quad \therefore \ln r = \ln \omega + \ln C$$

$$\ln r = \ln \omega + \ln C$$

Thus, we have shown that the velocity of a particle in a circular orbit is proportional to the radius.

From (1) it is seen that $\frac{d}{dt} \left(\frac{1}{r} \right) = -\frac{1}{r^2} \frac{dr}{dt}$

$$\frac{d}{dt} \left(\frac{1}{r} \right) = -\frac{1}{r^2} \frac{dr}{dt}$$

$$M_{O2} = (-)\frac{S_2}{\alpha_2} = (-)\frac{\int_{\phi_{1p}}^A \left(\frac{Mds}{EI}\right)_2}{L_2}, \quad M_{O1} = (-)\frac{S_1}{\alpha_1} = (-)\frac{\int_{\phi_{1p}}^A \left(\frac{Mds}{EI}\right)_1}{L_1}$$

$$\therefore \frac{(M_O)_2}{(M_O)_1} = \frac{\frac{\int_{\phi_{1p}}^A \left(\frac{Mds}{EI}\right)_1 \cdot \left(\frac{S_2}{S_1}\right)^2}{L_1 \cdot \left(\frac{S_2}{S_1}\right)}}{\frac{\int_{\phi_{1p}}^A \left(\frac{Mds}{EI}\right)_1}{L_1}} = \frac{S_2}{S_1}$$

$$(M_O)_2 = (M_O)_1 \frac{S_2}{S_1} \quad (M_O's \text{ vary directly as Span})$$

Since the Moment Areas of $(V_O)_2$ and $(M_O)_2$ to any point in Arch 2 are $\frac{S_2}{S_1}$ times the moment areas of $(V_O)_1$ and $(M_O)_1$ to the corresponding point in Arch 1, and since $(M_O)_2 = (M_O)_1 \left(\frac{S_2}{S_1}\right)$, then it is proved that the fixed end moment of Arch 2 is equal to the fixed end moment of Arch 1 times $\frac{S_2}{S_1}$ as illustrated below.



Taking moments at left springing

$$(M_L)_1 = (M_O)_1 + (V_O)_1 x_1 + (H_O)_1 Y_1 \quad (M_L)_2 = (M_O)_1 \left(\frac{S_2}{S_1}\right) + (V_O)_1 x_1 \left(\frac{S_2}{S_1}\right) + (H_O)_1 Y_1 \left(\frac{S_2}{S_1}\right)$$

$$\frac{(M_L)_2}{(M_L)_1} = \frac{S_2}{S_1} \quad \text{or} \quad (M_L)_2 = (M_L)_1 \frac{S_2}{S_1}$$

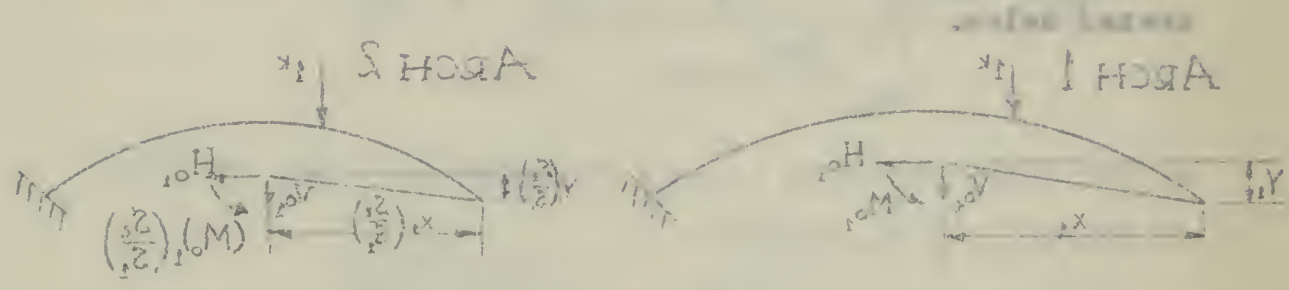
or Fixed End Moment varies directly as the Span.

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma$$

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma$$

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma$$

... of ...



$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma$$

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma$$

... of ...

VI. DETERMINATION OF CARRY-OVER FACTOR

When an arch is cut at the left springing and redundant forces are applied as shown below, deflections of the left springing will be as follows:

$$(1) \Delta V = H_a \delta'_{vm} + V_a \delta_{vv} + H_a \delta_{vh}$$

$$(2) \Delta H = H_a \delta'_{hm} + V_a \delta_{hv} + H_a \delta_{hh}$$

$$(3) \theta = H_a \alpha'_{\theta m} + V_a \alpha'_{\theta v} + H_a \alpha'_{\theta h}$$



By multiplying both sides of equations (1), (2) and (3) by EI , these equations take the form:

$$(4) EI \Delta V = H_a (EI \delta'_{vm}) + V_a (EI \delta_{vv}) + H_a (EI \delta_{vh})$$

$$(5) EI \Delta H = H_a (EI \delta'_{hm}) + V_a (EI \delta_{hv}) + H_a (EI \delta_{hh})$$

$$(6) EI \theta = H_a (EI \alpha'_{\theta m}) + V_a (EI \alpha'_{\theta v}) + H_a (EI \alpha'_{\theta h})$$

From the theories of virtual work and Maxwell's Law of Reciprocal Deflections:

$$EI \delta'_{vm} = EI \alpha'_{\theta v} = \int x ds = \widehat{L} \cdot \frac{S}{2}$$

$$EI \delta'_{hm} = EI \alpha'_{\theta h} = \int y ds = \widehat{L} \cdot [\widehat{R} \cos - (R - \bar{y})]$$

$$EI \alpha'_{\theta m} = \int ds = \widehat{L}$$

$$EI \delta_{vh} = EI \delta_{hv} = \int xy ds = \widehat{L} \cdot \frac{S}{2} \cdot [\widehat{R} \cos - (R - \bar{y})]$$

... and ...

... and ...

... and ...

$$\delta_1 + \delta_2 + \delta_3 = V\Delta \quad (1)$$

$$\delta_1 + \delta_2 + \delta_3 = H\Delta \quad (2)$$

$$\alpha_1 + \alpha_2 + \alpha_3 = e \quad (3)$$



... and ...

... and ...

$$\delta_1 + \delta_2 + \delta_3 = V\Delta \quad (1)$$

$$\delta_1 + \delta_2 + \delta_3 = H\Delta \quad (2)$$

$$\alpha_1 + \alpha_2 + \alpha_3 = e \quad (3)$$

... and ...

... and ...

$$\delta_1 + \delta_2 + \delta_3 = V\Delta \quad (1)$$

$$\delta_1 + \delta_2 + \delta_3 = H\Delta \quad (2)$$

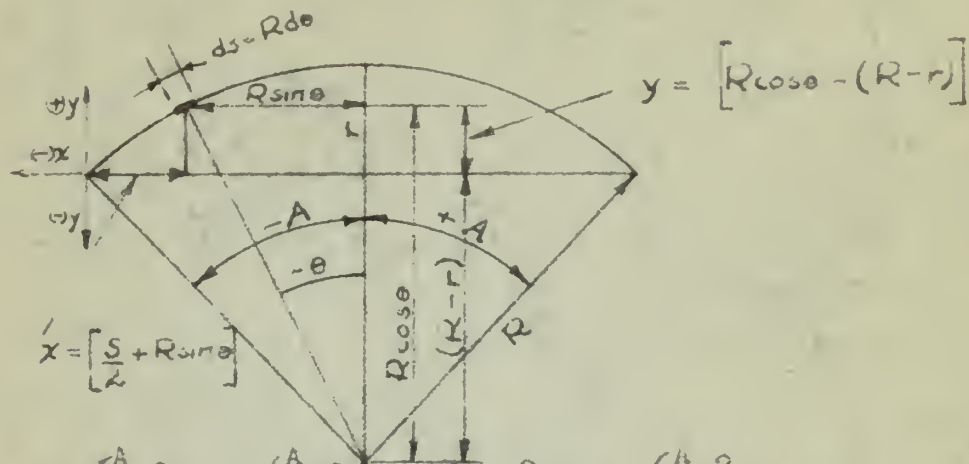
$$\alpha_1 + \alpha_2 + \alpha_3 = e \quad (3)$$

$$\delta_1 + \delta_2 + \delta_3 = V\Delta \quad (1)$$

$$EI \delta_{vv} = \int x^2 ds$$

$$EI \delta_{hh} = \int y^2 ds$$

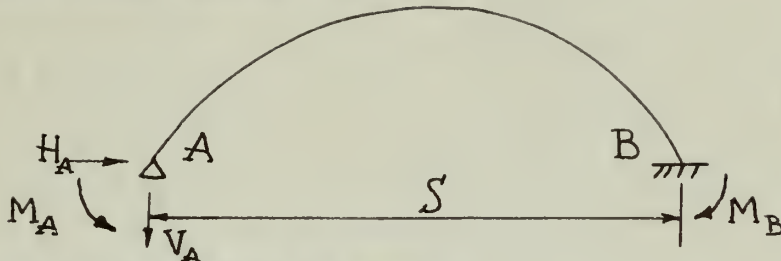
From the figure below, the right sides of the above equations are integrated such that $EI \delta_{vv}$ and $EI \delta_{hh}$ are expressed in terms of Rise, Span, Radius and Arch length.



$$\begin{aligned} EI \delta_{vv} &= \int_{-A}^A x^2 ds = \int_{-A}^A \left(\frac{S}{2} + R \sin \theta \right)^2 R d\theta = \int_{-A}^A \left(\frac{S^2}{4} + SR \sin \theta + R^2 \sin^2 \theta \right) R d\theta \\ &= \frac{RS^2}{4} \int_{-A}^A d\theta + SR^2 \int_{-A}^A \sin \theta d\theta + R^3 \int_{-A}^A \sin^2 \theta d\theta \\ &= \frac{RS^2}{4} \left[\theta \right]_{-A}^A - SR^2 \left[\cos \theta \right]_{-A}^A + R^3 \left[\frac{\theta}{2} - \frac{\sin \theta \cos \theta}{2} \right]_{-A}^A \\ &= \frac{RS^2}{4} (2) \frac{\widehat{L}}{2R} - 0 + \frac{R^3}{2} \left[\frac{\widehat{L}}{2R} - \frac{S}{2R} \left(\frac{R-r}{R} \right) + \frac{\widehat{L}}{2R} - \frac{S}{2R} \left(\frac{R-r}{R} \right) \right] \\ &= \frac{S^2 \widehat{L}}{4} + \frac{R}{2} [\widehat{L}R - S(R-r)] \end{aligned}$$

$$\begin{aligned} EI \delta_{hh} &= \int_{-A}^A y^2 ds = 2 \int_0^A [R \cos \theta - (R-r)]^2 R d\theta = 2 \int_0^A [R^2 \cos^2 \theta - 2(R-r)R \cos \theta + (R-r)^2] R d\theta \\ &= 2 \int_0^A R^3 \cos^2 \theta d\theta - 4 \int_0^A R^2 (R-r) \cos \theta d\theta + 2 \int_0^A R (R-r)^2 d\theta \\ &= 2R^3 \left[\frac{\theta}{2} + \frac{\sin \theta \cos \theta}{2} \right]_0^A - 4R^2 (R-r) [\sin \theta]_0^A + 2R (R-r)^2 [\theta]_0^A \\ &= R^3 \left[\frac{\widehat{L}}{2R} + \frac{S}{2R} \left(\frac{R-r}{R} \right) \right] - 4R^2 (R-r) \frac{S}{2R} + 2R (R-r)^2 \frac{\widehat{L}}{2R} \\ &= \frac{R}{2} [\widehat{L}R + S(R-r)] - 2R(R-r)S + (R-r)^2 \widehat{L} \end{aligned}$$

Since the carry-over factor is equal numerically to the moment induced at the fixed end of a member when a unit moment is applied at the hinged end, the carry-over factor for the arch may be obtained by equating equations (4) and (5) to zero and solving simultaneously for V_a and H_a in terms of M_a . Then by taking moments about the fixed end, M_b can be expressed as the carry-over factor times M_a .



Equating (4) and (5) to zero

$$(4') \quad 0 = EI [M_a (\delta'_{vm}) + V_a (\delta_{vv}) + H_a (\delta_{vh})]$$

$$(5') \quad 0 = EI [M_a (\delta'_{hm}) + V_a (\delta_{hv}) + H_a (\delta_{hh})]$$

Multiplying (4') by δ_{hh} and (5') by $(-)\delta_{vh}$, and dividing (4') and (5') by EI

$$(-) V_a (\delta_{vv}) (\delta_{hh}) = M_a (\delta'_{vm}) (\delta_{hh}) + H_a \delta_{vh} \delta_{hh}$$

$$(+) V_a (\delta_{hv}) (\delta_{vh}) = (-) M_a (\delta'_{hm}) (\delta_{vh}) (-) H_a \delta_{hh} \delta_{vh}$$

$$V_a = \frac{(EI)^2 \left[\frac{(\delta'_{vm})(\delta_{hh})}{(\delta_{hv})(\delta_{vh})} - \frac{(\delta'_{hm})(\delta_{vh})}{(\delta_{vv})(\delta_{hh})} \right]}{(EI)^2} M_a$$

For sake of brevity, we write the above equation

$$\text{as } V_a = (C_2) M_a$$

Multiplying (4') by δ_{hv} and (5') by $(-)\delta_{vv}$

$$(-) M_a (\delta_{vh}) (\delta_{hv}) = M_a (\delta'_{vm}) (\delta_{hv}) + V_a (\delta_{vv}) (\delta_{hv})$$

$$(+) M_a (\delta_{hh}) (\delta_{vv}) = (-) M_a (\delta'_{hm}) (\delta_{vv}) (-) V_a (\delta_{hv}) (\delta_{vv})$$

$$M_a = \frac{(EI)^2 \left[\frac{(\delta'_{vm})(\delta_{hv})}{(\delta_{hh})(\delta_{vv})} - \frac{(\delta'_{hm})(\delta_{vv})}{(\delta_{vh})(\delta_{hv})} \right]}{(EI)^2} M_a$$

We write the above as: $M_a = (C_1) M_a$

Since the support reaction is not completely in the
 vertical direction it is found that a fixed end is not
 required at the right end, but rather a roller support
 will be sufficient to maintain equilibrium for the beam.
 The deflection at the right end is found to be zero.
 The support reaction is found to be



$$\begin{aligned} (1) \quad & \left[\frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) + \frac{\partial}{\partial z} \right] \phi = 0 \\ (2) \quad & \left[\frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) + \frac{\partial}{\partial z} \right] \psi = 0 \end{aligned}$$

$$\begin{aligned} (3) \quad & \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \phi = \frac{\partial}{\partial z} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \psi \\ (4) \quad & \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \psi = \frac{\partial}{\partial z} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \phi \end{aligned}$$

$$\phi = \frac{\left[\frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) - \frac{\partial}{\partial z} \right] \psi}{\left[\frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) + \frac{\partial}{\partial z} \right]}$$

The boundary conditions are

$$\phi = \psi = 0 \text{ at } z = 0$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \psi}{\partial z} = 0 \text{ at } z = 2l$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \phi = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \psi = 0 \text{ at } x = 0$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \psi = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \phi = 0 \text{ at } x = 2l$$

$$\phi = \frac{\left[\frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) - \frac{\partial}{\partial z} \right] \psi}{\left[\frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) + \frac{\partial}{\partial z} \right]}$$

$$\phi = \psi = 0 \text{ at } z = 0$$

Taking moments at the fixed end

$$\begin{aligned} M_b &= M_a + V_a S \\ &= M_a + (D_2) M_a S = (1 + D_2 S) (M_a) \end{aligned}$$

The induced moment at the fixed end

$$-M_b = (-)(1 + D_2)(S) M_a$$

$$\underline{\text{The Carry-Over Factor} = \underline{(-)(1 + D_2)(S)}}$$

Although the term M_a does not appear in the moment equation above, this thrust must be considered in the solution of an Arched Bent by moment distribution since every time a joint is rotated and the moments balanced out at the joint, the horizontal restraining force at the ends of the arch resulting from the applied moments must be determined. By keeping a record of the applied moments, the H.R.F. may be determined by $H_a = (D_1) M_a$.

VI.(a) ELIMINATION OF RELATIONSHIP BETWEEN CARRY-OVER FACTOR AND SPAN FOR THE SAME RISE/SPAN RATIO.

Since the deflection condition equations (1), (2) and (3) are applicable to an arch of any span, it is necessary only to determine how the various deflections vary with span in order to find the relationship between carry-over factor and span.

From the fact that arches of the same Rise to Span ratio are similar as previously proved, the deflections for an arch of Span S_2 are written in terms of the deflections of an arch of Span S_1 .

General

For Arch #1

For Arch #2

$$\begin{aligned}
 EI\delta'_{vm} &= EI\alpha'_{vm} = \int x ds & \int (X_1) ds_1 & \int (X_1 \cdot \frac{s_2}{s_1}) ds_1 \cdot \frac{s_2}{s_1} = (\frac{s_2}{s_1})^2 \int (X_1) ds_1 \\
 EI\delta'_{hm} &= EI\alpha'_{hm} = \int y ds & \int (Y_1) ds_1 & \int (Y_1 \cdot \frac{s_2}{s_1}) ds_1 \cdot \frac{s_2}{s_1} = (\frac{s_2}{s_1})^2 \int (Y_1) ds_1 \\
 EI\alpha_{vm} &= \int ds & \int ds_1 & \int ds_1 \cdot \frac{s_2}{s_1} = (\frac{s_2}{s_1}) \int ds_1 \\
 EI\delta_{vh} &= EI\delta_{hv} = \int xy ds & \int X_1 Y_1 ds_1 & \int (X_1 \cdot \frac{s_2}{s_1}) (Y_1 \cdot \frac{s_2}{s_1}) ds_1 \cdot \frac{s_2}{s_1} = (\frac{s_2}{s_1})^3 \int ds_1 \\
 EI\delta_{vv} &= \int x^2 ds & \int (X_1)^2 ds_1 & \int (X_1 \cdot \frac{s_2}{s_1})^2 ds_1 \cdot \frac{s_2}{s_1} = (\frac{s_2}{s_1})^3 \int (X_1)^2 ds_1 \\
 EI\delta_{hh} &= \int y^2 ds & \int (Y_1)^2 ds_1 & \int (Y_1 \cdot \frac{s_2}{s_1})^2 ds_1 \cdot \frac{s_2}{s_1} = (\frac{s_2}{s_1})^3 \int (Y_1)^2 ds_1
 \end{aligned}$$

Substituting the above deflections for Point A of Arch 2, for the corresponding deflections for Arch 1, we obtain

$$\begin{aligned}
 V_a &= D_2 \frac{(\frac{s_2}{s_1})^5}{(\frac{s_2}{s_1})^6} M_a = D_2 (\frac{s_1}{s_2}) M_a \\
 H_a &= D_1 \frac{(\frac{s_2}{s_1})^5}{(\frac{s_2}{s_1})^6} M_a = D_1 (\frac{s_1}{s_2}) M_a
 \end{aligned}$$

Taking Moments at the fixed end of Arch 2 of Span $s_2 =$

$$\begin{aligned}
 & s_1 (\frac{s_2}{s_1}) \\
 M_b &= M_a + V_a s_2 = M_a + D_2 (\frac{s_1}{s_2}) M_a s_2 = M_a + D_2 (\frac{s_1}{s_2}) M_a s_1 (\frac{s_2}{s_1}) \\
 M_b &= M_a + D_2 M_a s_1 = (1 + D_2 s_1) M_a
 \end{aligned}$$

The induced moment at the fixed end

$$-M_b = -(1 + D_2 s_1) M_a$$

The Carry-Over Factor for Arch 2 = $-(1 + D_2 s_1)$ which is the same as the C.O.F. for Arch 1.

∴ Carry-Over Factor for Arches of same Rise/Span ratio is independent of span.

2nd Edn

1. 2

1870

2

1. 2

207

1000

VII DETERMINATION OF ABSOLUTE STIFFNESS

Since the absolute stiffness is defined as the moment required to produce unit rotation (1 radian) of the hinged end of a member whose far end is rigidly fixed, the absolute stiffness for the arch may be obtained by setting θ of equation (6) equal to 1, and substituting the values of V_a and H_a in terms of M_a (previously determined) in the right side of the equation. By solving the resulting equation for M_a , the absolute stiffness is determined.

$$(6) \quad EI(\theta) = M_a(EI\alpha_{\theta m}) + V_a(EI\alpha'_{\theta v}) + H_a(EI\alpha'_{\theta h})$$

After substitution:

$$EI(1) = M_a(EI\alpha_{\theta m}) + (D_2)M_a(EI\alpha'_{\theta v}) + (D_1)M_a(EI\alpha'_{\theta h})$$

$$M_a = \frac{EI}{EI\alpha_{\theta m} + (D_2)EI\alpha'_{\theta v} + (D_1)EI\alpha'_{\theta h}}$$

which, for sake of brevity, is written

$$M_a = \frac{EI}{D_3} = \text{Absolute Stiffness}$$

VIII DETERMINATION OF RELATIONSHIP BETWEEN ABSOLUTE STIFFNESS AND SPAN FOR THE CASE RISE/SPAN RATIO.

Using the relationship between deflections and span that have been previously determined, we write equation (6) for an arch of span S_2

$$EI(\theta) = M_a \left[EI(\alpha_{\theta m} \frac{S_2}{S_1}) \right] + V_a \left[EI(\alpha'_{\theta v}) (\frac{S_2}{S_1})^2 \right] + H_a \left[EI(\alpha'_{\theta h}) (\frac{S_2}{S_1})^2 \right]$$

...the ... of ...
...the ... of ...
...the ... of ...
...the ... of ...
...the ... of ...
...the ... of ...
...the ... of ...
...the ... of ...
...the ... of ...
...the ... of ...

$$(\frac{1}{2}x^2 + \frac{1}{2}x^2) + (\frac{1}{2}x^2 + \frac{1}{2}x^2) + (\frac{1}{2}x^2 + \frac{1}{2}x^2) = 3x^2$$

...the ... of ...

$$(\frac{1}{2}x^2 + \frac{1}{2}x^2) + (\frac{1}{2}x^2 + \frac{1}{2}x^2) + (\frac{1}{2}x^2 + \frac{1}{2}x^2) = 3x^2$$

$$\frac{1}{2}x^2 + \frac{1}{2}x^2 = x^2$$

...the ... of ...

$$\frac{1}{2}x^2 + \frac{1}{2}x^2 = x^2$$

...the ... of ...

...the ... of ...

$$\left[\left(\frac{1}{2}x^2 + \frac{1}{2}x^2 \right) + \left(\frac{1}{2}x^2 + \frac{1}{2}x^2 \right) + \left(\frac{1}{2}x^2 + \frac{1}{2}x^2 \right) \right] = 3x^2$$

Substituting the values of V_a and H_a as previously determined for an arch of span S_2

$$EI(1) = H_a \left[EI(\alpha_{em} \frac{S_2}{S_1}) \right] + D_2 \left(\frac{S_1}{S_2} \right) H_a \left[EI\alpha'_{ev} \left(\frac{S_2}{S_1} \right)^2 \right] + D_1 \left(\frac{S_1}{S_2} \right) H_a \left[EI\alpha'_{eh} \left(\frac{S_2}{S_1} \right)^2 \right]$$

$$EI(1) = H_a \left[EI\alpha_{em} \left(\frac{S_2}{S_1} \right) \right] + D_2 H_a \left[EI\alpha'_{ev} \left(\frac{S_2}{S_1} \right) \right] + D_1 H_a \left[EI\alpha'_{eh} \left(\frac{S_2}{S_1} \right) \right]$$

$$H_a = \frac{EI}{[EI\alpha_{em} + (D_2)EI\alpha'_{ev} + D_1EI\alpha'_{eh}] \frac{S_2}{S_1}}$$

which is written $H_a = \frac{EI(S_1)}{D_3 S_2}$

∴ Absolute stiffness varies inversely as the span.

VIII DETERMINATION OF THE INDUCED MOMENT AND REACTIONS (VERTICAL AND HORIZONTAL) OCCURRING AT THE END OF AN ARCH DUE TO SPREADING OF THE ARCH UNDER A LOAD.

Every vertical load placed on an arch produces a tendency for the arch to spread. Since the column members of an arched bent do not have infinite stiffness, a definite spreading of the arch will occur under the load. This spread must be taken into consideration when solving an arched bent by moment distribution.

To determine the reactions in terms of spread, we subject the left end of the arch to a spread of ΔH feet (allowing no vertical or rotational deflection of that end)



By equating equations (4) and (6) to zero and solving equations (4), (5) and (6) simultaneously in terms of $EI\Delta H$,

Consider the system of two particles

of masses m_1 and m_2 moving with velocities \mathbf{v}_1 and \mathbf{v}_2

$$(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{v} \quad (1)$$

$$(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{v} \quad (2)$$

$$\mathbf{v} = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2}$$

$$\mathbf{v} = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2}$$

the velocity of the center of mass is

$$\mathbf{v} = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2}$$

the velocity of the center of mass is the same as the velocity of the system as a whole. This is true because the center of mass is the point at which the total momentum of the system is zero. If the system is moving with a velocity \mathbf{v} , then the center of mass is also moving with a velocity \mathbf{v} . This is true because the center of mass is the point at which the total momentum of the system is zero. If the system is moving with a velocity \mathbf{v} , then the center of mass is also moving with a velocity \mathbf{v} .



the height difference ΔH is the same as the height difference ΔH between the two points A and B.

we obtain the desired reactions.

$$(a) \quad 0 = M_a(EI \delta'_{vm}) + V_a(EI \delta_{vv}) + H_a(EI \delta_{vh})$$

$$(b) \quad 0 = M_a(EI \alpha'_{\theta m}) + V_a(EI \alpha'_{\theta v}) + H_a(EI \alpha'_{\theta h})$$

Determination of V_a as caused by spread ΔH

By multiplying the above equations by $\alpha'_{\theta h}$ and $(-) \delta_{vh}$ respectively.

$$-V_a(EI \delta_{vv}) \alpha'_{\theta h} = M_a(EI \delta'_{vm}) \alpha'_{\theta h} + H_a(EI \delta_{vh}) \alpha'_{\theta h}$$

$$V_a(EI \alpha'_{\theta v}) \delta_{vh} = (-) M_a(EI \alpha_{\theta m}) \delta_{vh} (-) H_a(EI \alpha'_{\theta h}) \delta_{vh}$$

$$V_a = \frac{(EI)^2}{(EI)^2} \left[\frac{(\delta'_{vm})(\alpha'_{\theta h}) - (\alpha_{\theta m})(\delta_{vh})}{(\alpha'_{\theta v})(\delta_{vh}) - (\delta_{vv})(\alpha'_{\theta h})} \right] M_a$$

Substituting values previously obtained for $EI \delta'_{vm}$, $EI \alpha'_{\theta h}$, etc.

$$V_a = \frac{\left[\left(\frac{L}{2} \right) \left[L \cdot \left[\text{Rise} - (R - \bar{y}) \right] \right] - \left(\frac{L}{2} \right) \left[\left(\frac{L}{2} \right) \cdot \left[\text{Rise} - (R - \bar{y}) \right] \right] \right]}{\left[\left(\frac{L}{2} \right) \left[L \cdot \left[\text{Rise} - (R - \bar{y}) \right] \right] - \left[x^2 ds \right] \left[L - \left[\text{Rise} - (R - \bar{y}) \right] \right] \right]} M_a$$

Since the numerator of the above expression is equal to zero, then $V_a = 0$

Determination of M_a as caused by spread ΔH

Substituting $V_a = 0$ in equation A

$$\text{we obtain } H_a = (-) M_a \frac{(\delta'_{vm})}{(\delta_{vh})}$$

$$\text{Substituting } V_a = 0 \text{ and } H_a = (-) M_a \frac{(\delta'_{vm})}{(\delta_{vh})}$$

in equation (5), we obtain

$$EI \Delta H = EI \left[M_a(\delta'_{hm}) - M_a \frac{(\delta'_{vm})}{(\delta_{vh})} (\delta_{hh}) \right]$$

$$M_a = \frac{EI \Delta H}{EI \delta'_{hm} - \frac{(\delta'_{vm})(\delta_{hh})(EI)^2}{EI \delta_{vh}}} = \frac{EI \Delta H}{D_4}$$

$$1. \delta \alpha_1 + \delta \alpha_2 + \delta \alpha_3 = 0$$

$$1. \alpha_1 \delta + \alpha_2 \delta + \alpha_3 \delta = 0$$

HA ...

$$1. \alpha_1 \delta + \alpha_2 \delta + \alpha_3 \delta = 0$$

$$1. \alpha_1 \delta + \alpha_2 \delta + \alpha_3 \delta = 0$$

$$1. \alpha_1 \delta + \alpha_2 \delta + \alpha_3 \delta = 0$$

$$1. \alpha_1 \delta + \alpha_2 \delta + \alpha_3 \delta = 0$$

...

$$1. \alpha_1 \delta + \alpha_2 \delta + \alpha_3 \delta = 0$$

$$\epsilon^M \left[\frac{[\dots]}{[\dots]} \right]$$

...

$$1. \alpha_1 \delta + \alpha_2 \delta + \alpha_3 \delta = 0$$

HA ...

...

$$1. \alpha_1 \delta + \alpha_2 \delta + \alpha_3 \delta = 0$$

$$1. \alpha_1 \delta + \alpha_2 \delta + \alpha_3 \delta = 0$$

...

$$1. \alpha_1 \delta + \alpha_2 \delta + \alpha_3 \delta = 0$$

$$1. \alpha_1 \delta + \alpha_2 \delta + \alpha_3 \delta = 0$$

Determination of M_a as caused by spread ΔH

Substituting $V_a = 0$ and $M_a = (-)\frac{H_a(\delta_{vh})}{(\delta'_{vm})}$ in (5)

$$EI \Delta H = EI (-)\frac{H_a(\delta_{vh})}{(\delta'_{vm})} (\delta'_{hm}) + H_a(\delta_{hh})$$

$$H_a = \frac{EI \Delta H}{EI \delta_{hh} - \frac{(\delta_{vh})(\delta'_{hm})(EI)^2}{EI \delta'_{vm}}} = \frac{EI \Delta H}{L_5}$$

VIII(a) DETERMINATION OF THE RELATIONSHIP BETWEEN THE REACTIONS, INDUCED AS A RESULT OF SPREAD, AND THE SPAN FOR THE SAME RISE/SPAN RATIO.

Using the relationship between deflections and span that have been previously determined, and substituting in the appropriate equations, we obtain,

Relationship between V_a and Span for Arch 2

$$V_a = \frac{(EI)^2}{(EI)^2} \left[\frac{(\delta'_{vm})(\frac{s_2}{s_1})^2(\alpha'_{eh})(\frac{s_2}{s_1})^2 - (\alpha_{em})(\frac{s_2}{s_1})(\delta_{vh})(\frac{s_2}{s_1})^3}{(\alpha'_{ev})(\frac{s_2}{s_1})^2(\delta_{vh})(\frac{s_2}{s_1})^3 - (\delta_{vv})(\frac{s_2}{s_1})^3(\alpha'_{eh})(\frac{s_2}{s_1})^2} \right] M_a$$

$$= \left[\frac{(\delta'_{vm})(\alpha'_{eh}) - (\alpha_{em})(\delta_{vh})}{(\alpha'_{ev})(\delta_{vh}) - (\delta_{vv})(\alpha'_{eh})} \right] \frac{(\frac{s_2}{s_1})^4}{(\frac{s_2}{s_1})^5} M_a$$

Since the numerator of the term in the bracket has been found equal to zero, we find

$$\underline{V_a = 0 \quad (\text{regardless of Span})}$$

Relationship between M_a and Span for Arch 2

$$M_a = \frac{EI \Delta H}{EI \delta'_{hm}(\frac{s_2}{s_1})^2 - (\delta'_{vm})(\frac{s_2}{s_1})^2(\delta_{hh})(\frac{s_2}{s_1})^3(EI)^2}$$

$$\frac{EI(\delta_{vh})(\frac{s_2}{s_1})^3}{EI(\delta_{vh})(\frac{s_2}{s_1})^3}$$

HA Energy of transition from the initial state

for $\delta = \frac{1}{2} \frac{\pi}{\omega} \omega_0$ is, for the $0 \rightarrow 1$ transition

$$E_{0 \rightarrow 1} = \frac{1}{2} \frac{\pi}{\omega} \omega_0 \left(\frac{1}{2} \frac{\pi}{\omega} \omega_0 \right) = \frac{1}{4} \frac{\pi^2}{\omega^2} \omega_0^2$$

$$\frac{HA}{E_{0 \rightarrow 1}} = \frac{HA}{\frac{1}{4} \frac{\pi^2}{\omega^2} \omega_0^2} = \frac{4}{\pi^2} \frac{HA \omega^2}{\omega_0^2}$$

The above expression is a function of ω/ω_0
and ω_0 is the natural frequency
of the oscillator

For the purpose of the present calculation we shall
 assume that the oscillator is in the ground state
 and that the energy of the photon is $h\nu$

Let us now consider the case of a harmonic oscillator

$$S_M \left[\frac{\frac{1}{2} \frac{\pi}{\omega} \omega_0 \left(\frac{1}{2} \frac{\pi}{\omega} \omega_0 \right) - \frac{1}{2} \frac{\pi}{\omega} \omega_0 \left(\frac{1}{2} \frac{\pi}{\omega} \omega_0 \right)}{\frac{1}{2} \frac{\pi}{\omega} \omega_0 \left(\frac{1}{2} \frac{\pi}{\omega} \omega_0 \right) - \frac{1}{2} \frac{\pi}{\omega} \omega_0 \left(\frac{1}{2} \frac{\pi}{\omega} \omega_0 \right)} \right] \frac{1}{2} \frac{\pi}{\omega} \omega_0$$

$$= \frac{1}{2} \frac{\pi}{\omega} \omega_0 \left[\frac{\frac{1}{2} \frac{\pi}{\omega} \omega_0 \left(\frac{1}{2} \frac{\pi}{\omega} \omega_0 \right) - \frac{1}{2} \frac{\pi}{\omega} \omega_0 \left(\frac{1}{2} \frac{\pi}{\omega} \omega_0 \right)}{\frac{1}{2} \frac{\pi}{\omega} \omega_0 \left(\frac{1}{2} \frac{\pi}{\omega} \omega_0 \right) - \frac{1}{2} \frac{\pi}{\omega} \omega_0 \left(\frac{1}{2} \frac{\pi}{\omega} \omega_0 \right)} \right]$$

For the purpose of the present calculation we shall
 assume that the oscillator is in the ground state
 and that the energy of the photon is $h\nu$

Let us now consider the case of a harmonic oscillator

$$\frac{HA}{E_{0 \rightarrow 1}} = \frac{HA}{\frac{1}{4} \frac{\pi^2}{\omega^2} \omega_0^2} = \frac{4}{\pi^2} \frac{HA \omega^2}{\omega_0^2}$$

$$M_a = \frac{EI \Delta H}{\left[EI \delta'_{hm} - \frac{(\delta'_{vm})(\delta_{hh})(EI)^2}{EI (\delta_{vh})} \right] \left(\frac{s_2}{s_1} \right)^2}$$

$$= \frac{EI \Delta H (s_1)^3}{D_4 (s_2)^2}$$

$\therefore M_a$ varies inversely as the square of the Span.

Relationship between H_a and Span for Arch 2

$$H_a = \frac{EI \Delta H}{\frac{EI \delta_{hh} \left(\frac{s_2}{s_1} \right)^3 - (\delta_{vh}) \left(\frac{s_2}{s_1} \right)^3 (\delta'_{hm}) \left(\frac{s_2}{s_1} \right)^2 (EI)^2}{EI \delta'_{vm} \left(\frac{s_2}{s_1} \right)^2}}$$

$$= \frac{EI \Delta H}{\left[EI \delta_{hh} - \frac{(\delta_{vh})(\delta'_{hm})(EI)^2}{EI \delta'_{vm}} \right] \left(\frac{s_2}{s_1} \right)^3}$$

$$= \frac{EI \Delta H (s_1)^3}{D_5 (s_2)^3}$$

$\therefore H_a$ varies inversely as the cube of the Span.

$$\frac{H\Delta}{2} \left[\frac{2\delta_1 + \delta_2 - \delta_3}{2} \right]$$

$$\frac{H\Delta}{2}$$

which will be the same as the original value \therefore

it will be the same as the original value

$$\frac{H\Delta}{2} \left[\frac{2\delta_1 + \delta_2 - \delta_3}{2} \right]$$

$$\frac{H\Delta}{2} \left[\frac{2\delta_1 + \delta_2 - \delta_3}{2} \right]$$

$$\frac{H\Delta}{2}$$

which will be the same as the original value \therefore

DEVELOPMENT OF METHOD FOR NON-PRISMATIC ARCH

$$I_x = I_c \cos \theta$$

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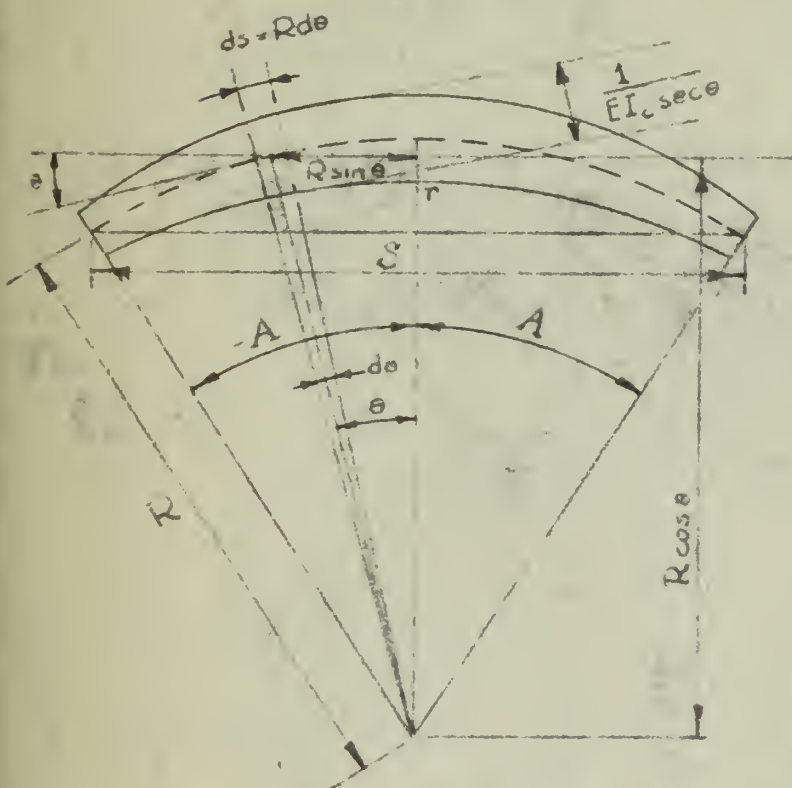
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DERIVATIONS FOR FIXED END MOMENT, THRUST, AND SHEAR FOR A NON-PRISMATIC CIRCULAR ARCH, THE I AT ANY POINT BEING EQUAL TO THE I AT THE CROWN (I_c) TIMES THE SECANT OF THE ANGLE FORMED BY A TANGENT TO THE CENTERLINE OF THE ARCH AT THE POINT IN QUESTION AND A LINE PARALLEL TO THE SPRINGING LINE.

I. Radius in terms of Rise and Span

As derived for a Prismatic Arch: $R = \frac{4r^2 + s^2}{8r}$

II. Determination of the Centroid of the Non-Prismatic Arch Conjugate Structure (Plan View)



$$\begin{aligned} dA &= \frac{ds}{EI} = \frac{R d\theta}{EI_c \sec \theta} \\ \bar{x} &= \frac{\int x dA}{\int dA} \\ \bar{x} &= \frac{\int (R \sin \theta) \frac{R d\theta}{EI_c \sec \theta}}{\int \frac{R d\theta}{EI_c \sec \theta}} = \\ &= \frac{R \int \sin \theta \cos \theta d\theta}{\int \cos \theta d\theta} \\ &= \frac{R \left[\frac{\sin^2 \theta}{2} \right]_{-A}^A}{\left[\sin \theta \right]_{-A}^A} = 0 \end{aligned}$$

$$\begin{aligned} \bar{y} &= \frac{\int y dA}{\int dA} = \frac{\int R \cos \theta \frac{R d\theta}{EI_c \sec \theta}}{\int \frac{R d\theta}{EI_c \sec \theta}} = \frac{R \int \cos^2 \theta d\theta}{\int \cos \theta d\theta} \\ &= \frac{R \left[\frac{\theta + \sin \theta \cos \theta}{2} \right]_{-A}^A}{\left[\sin \theta \right]_{-A}^A} = \frac{R \left[\frac{L}{2R} + \frac{S}{2R} \left(\frac{R-r}{R} \right) \right]}{\left[\frac{S}{2R} - (-) \frac{S}{2R} \right]} - \frac{R \left[-\frac{L}{2R} + \left(-\frac{S}{2R} \right) \left(\frac{R-r}{R} \right) \right]}{\left[\frac{S}{2R} - (-) \frac{S}{2R} \right]} \end{aligned}$$

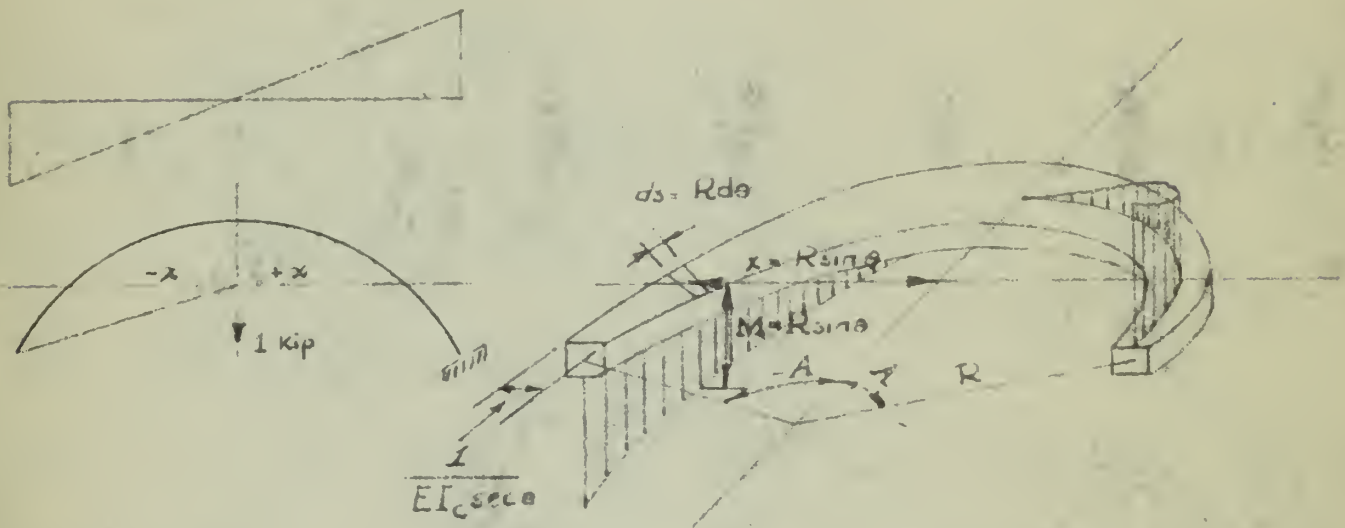
$$\bar{y} = \frac{R \left[\frac{LP + S(R-r)}{2R^2} \right]}{\frac{S}{R}} = \frac{LR + S(R-r)}{2S}$$

∴ For a symmetrical non-prismatic arch:

$$\bar{x} = 0 \quad \bar{y} = \frac{LR + S(R-r)}{2S}$$

III. DETERMINATION OF DEFLECTION AT NEUTRAL POINT DUE TO:

(A) 1 kip vertical force at N.P.

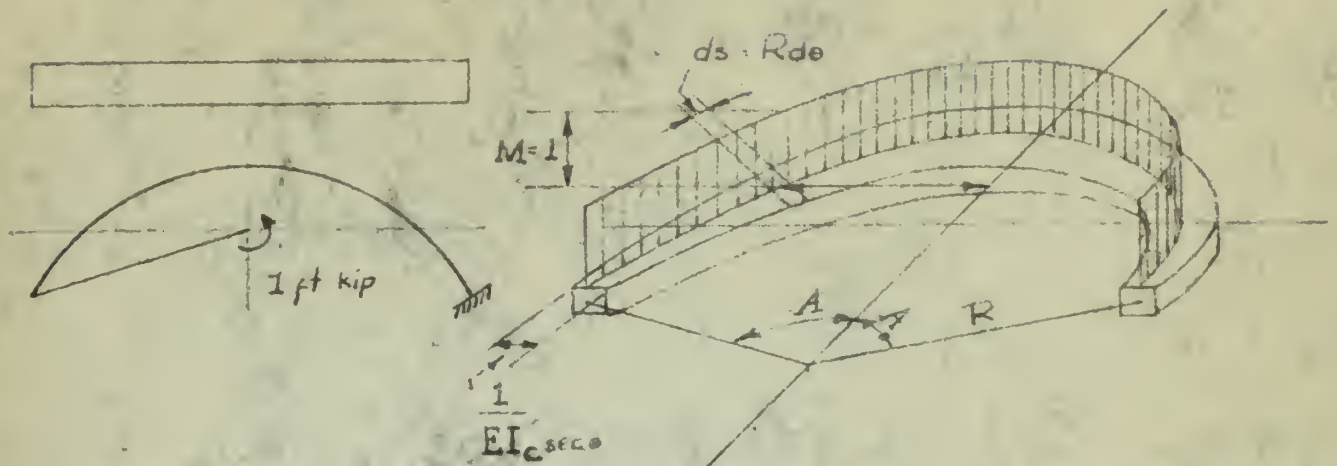


$$\delta_{vv} = \int_{-A}^A \frac{M ds(x)}{EI} =$$

$$EI_c \delta_{vv} = 2 \int_0^A (R \sin \theta) \left(\frac{R d\theta}{\sec \theta} \right) (R \sin \theta) = 2R^3 \int_0^A \sin^2 \theta \cos \theta d\theta$$

$$= 2R^3 \left[\frac{\sin^3 \theta}{3} \right]_0^A = \frac{2R^3}{3} \left[\frac{S}{2R} \right]^3 = \frac{2R^3 S^3}{(3)(8)R^3} = \frac{S^3}{12}$$

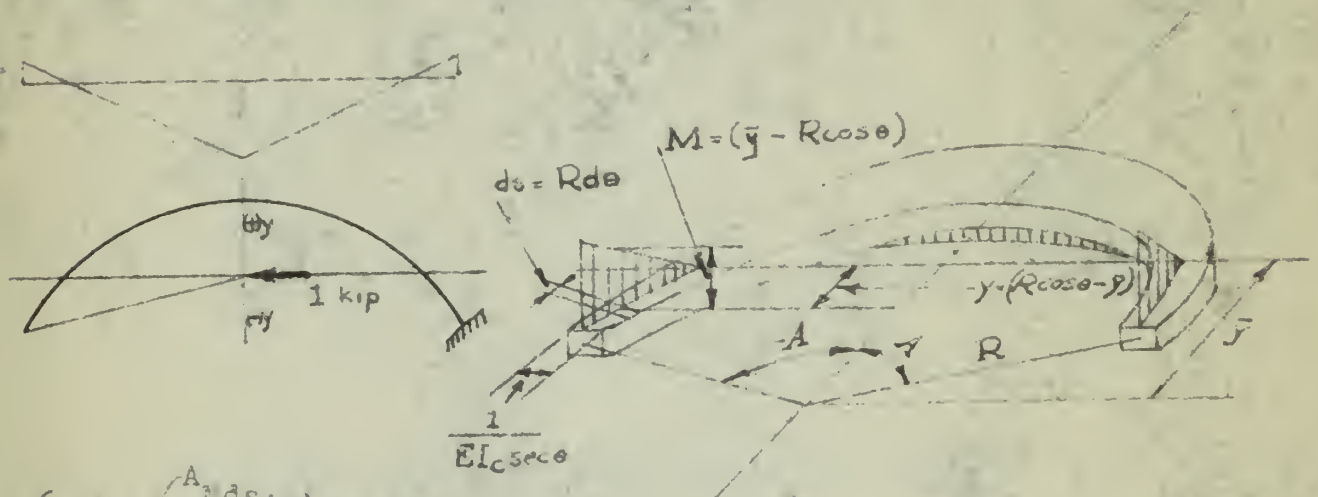
(B) 1 ft.kip couple at N.P.



$$\alpha_{em} = \int_{-A}^A \frac{M ds}{EI}$$

$$EI_c \alpha_{em} = \int_{-A}^A \frac{M(R d\theta)}{EI_c \sec \theta} = 2 \int_0^A (1) \frac{R d\theta}{\sec \theta} = 2 \int_0^A R \cos \theta d\theta = 2R [\sin \theta]_0^A = 2R \left[\frac{s}{2R} \right] = s$$

(c) 1 kip horizontal force at N.P.



$$\delta_{hh} = \int_{-A}^A \frac{M ds (y)}{EI}$$

$$\begin{aligned} EI_c \delta_{hh} &= 2 \int_0^A (\bar{y} - R \cos \theta) \frac{R d\theta}{\sec \theta} (R \cos \theta - \bar{y}) = 2 \int_0^A (\bar{y} - R \cos \theta) R \cos \theta d\theta (R \cos \theta - \bar{y}) \\ &= 2 \int_0^A [(-)(\bar{y})^2 \cos \theta + 2(\bar{y}) R \cos^2 \theta - R^2 \cos^3 \theta] R d\theta \\ &= 2 \left[(-)R(\bar{y})^2 \int_0^A \cos \theta d\theta + 2R^2(\bar{y}) \int_0^A \cos^2 \theta d\theta - R^3 \int_0^A \cos^3 \theta d\theta + R^3 \int_0^A \sin^2 \cos \theta d\theta \right] \\ &= 2 \left[(-)R(\bar{y})^2 [\sin \theta]_0^A + 2R^2 \bar{y} \left[\frac{\theta}{2} + \frac{\sin \theta \cos \theta}{2} \right]_0^A - R^3 [\sin \theta]_0^A + R^3 \left[\frac{\sin^3 \theta}{3} \right]_0^A \right] \end{aligned}$$

$$= 2 \left[\left[(-)R(\bar{y})^2 \frac{S}{2R} \right] + \frac{2P^2(\bar{y})}{2} \left[\frac{\bar{L}}{2R} + \frac{S}{2R} \frac{(P-r)}{R} \right] - R^3 \frac{S}{2R} + \frac{R^3}{3} \left(\frac{S}{2R} \right)^3 \right]$$

$$= -(\bar{y})^2 S + (\bar{y}) [\bar{L}R + S(R-r)] - R^2 S + \frac{S^3}{12}$$

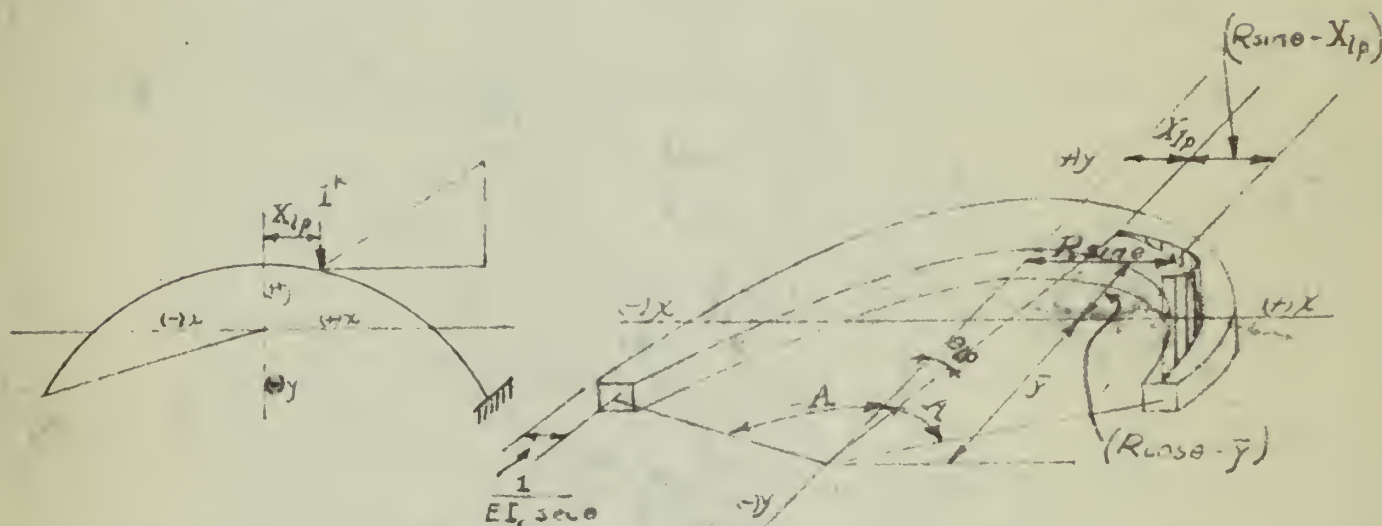
Since $\bar{y} = \frac{\bar{L}R + S(R-r)}{2S}$ we write the above as

$$= -(\bar{y})^2 S + (\bar{y})(\bar{y})(2S) - R^2 S + \frac{S^3}{12}$$

$$= -(\bar{y})^2 S + 2(\bar{y})^2 S - R^2 S + \frac{S^3}{12}$$

$$= (\bar{y})^2 S - R^2(S) + \frac{S^3}{12}$$

IV. DETERMINATION OF DEFLECTIONS AT THE NEUTRAL POINT (N.P.) CAUSED BY A 1 KIP VERTICAL LOAD ACTING ON THE ARCH AT A POINT THAT IS HORIZONTAL DISTANCE X_{lp} TO THE RIGHT OF THE CROWN.



(A) Determination of Vertical Deflection at N.P.

$$\Delta V = \int_{\theta_{lp}}^A \frac{Mds}{EI} (X)$$

$$EI_c \Delta V = \int_{\theta_{lp}}^A (R \sin \theta - X_{lp}) \left(\frac{Pd \theta}{\sec \theta} \right) (R \sin \theta) = \int_{\theta_{lp}}^A (R \sin \theta - X_{lp}) R \cos \theta d\theta (R \sin \theta)$$

$$= \int_{\theta_{lp}}^A R^3 \sin^2 \theta \cos \theta d\theta - \int_{\theta_{lp}}^A R^2 (X_{lp}) \sin \theta \cos \theta d\theta$$

$$= R^3 \left[\frac{\sin^3 \theta}{3} \right]_{\theta_{1p}}^{\theta_2} - R^2 (\chi_{1p}) \left[\frac{\sin^2 \theta}{2} \right]_{\theta_{1p}}^{\theta_2}$$

$$= \frac{R^3}{3} \left(\frac{R}{2R} \right)^3 - \frac{R^3}{3} (\sin^3 \theta_{1p}) - \frac{R^2}{2} (\chi_{1p}) \left(\frac{R}{2R} \right)^2 + \frac{R^2 (\chi_{1p})}{2} (\sin^2 \theta_{1p})$$

Since $\sin \theta_{1p} = \frac{\chi_{1p}}{R}$ we write the above as

$$= \frac{R^3}{24} - \frac{R^3}{3} \left(\frac{\chi_{1p}}{R} \right)^3 - \frac{(\chi_{1p})^2}{8} + \frac{R^2 (\chi_{1p})}{2} \left(\frac{\chi_{1p}}{R} \right)^2$$

$$= \frac{R^3}{24} - \frac{(\chi_{1p})^3}{3} - \frac{(\chi_{1p})^2}{8} + \frac{(\chi_{1p})^3}{2}$$

$$= \frac{R^3}{24} + \frac{(\chi_{1p})^3}{6} - \frac{(\chi_{1p})^2}{8}$$

(B) DETERMINATION OF HORIZONTAL DEFLECTION AT R.P.

$$\Delta H = \int_{\theta_{1p}}^{\theta_2} \frac{R d\theta (y)}{EI}$$

$$EI_c \Delta H = \int_{\theta_{1p}}^{\theta_2} (R \sin \theta - \chi_{1p}) \frac{R d\theta}{\sec \theta} (R \cos \theta - \bar{y}) = \int_{\theta_{1p}}^{\theta_2} (R \sin \theta - \chi_{1p}) R \cos \theta (R \cos \theta - \bar{y}) d\theta$$

$$= \int_{\theta_{1p}}^{\theta_2} R^3 \cos^2 \theta \sin \theta d\theta - \int_{\theta_{1p}}^{\theta_2} R^2 \bar{y} \sin \theta \cos \theta d\theta - \int_{\theta_{1p}}^{\theta_2} R^2 (\chi_{1p}) \cos^2 \theta d\theta$$

$$+ \int_{\theta_{1p}}^{\theta_2} R (\bar{y}) (\chi_{1p}) \cos \theta d\theta$$

$$= (-) R^3 \left[\frac{\cos^3 \theta}{3} \right]_{\theta_{1p}}^{\theta_2} - R^2 (\bar{y}) \left[\frac{\sin^2 \theta}{2} \right]_{\theta_{1p}}^{\theta_2} - R^2 (\chi_{1p}) \left[\frac{\theta + \sin \theta \cos \theta}{2} \right]_{\theta_{1p}}^{\theta_2}$$

$$+ R (\bar{y}) (\chi_{1p}) \left[\sin \theta \right]_{\theta_{1p}}^{\theta_2}$$

$$= (-) \frac{R^3}{3} \left[\left(\frac{R-r}{R} \right)^3 - \cos^3 \theta_{1p} \right] - \frac{R^2 (\bar{y})}{2} \left[\left(\frac{R}{2R} \right)^2 - \sin^2 \theta_{1p} \right] - \frac{R^2 \chi_{1p}}{2} \left[\frac{L}{2R} + \frac{3(R-r)}{2R} \right]$$

$$+ \frac{R^2 (\chi_{1p})}{2} \left[\theta_{1p} + \sin \theta_{1p} \cos \theta_{1p} \right] + R (\bar{y}) (\chi_{1p}) \left[\left(\frac{R}{2R} \right) - \sin \theta_{1p} \right]$$

$$= -\frac{(R-r)^3}{3} + \frac{R^3 \cos^3 \theta_{1p}}{3} - \frac{(\bar{y}) R^2}{2} + \frac{R^2 (\bar{y}) (\chi_{1p})^2}{2} - \frac{(\chi_{1p}) [L R + 3(R-r)]}{4}$$

$$\left[\frac{\partial^2 \phi}{\partial x^2} \right]_{x=0} = \left[\frac{\partial^2 \phi}{\partial x^2} \right]_{x=L}$$

$$(1 - \nu) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad (1)$$

$$\frac{\partial \phi}{\partial x} = 0 \quad \text{at } x = 0 \text{ and } x = L$$

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{at } y = 0 \text{ and } y = H$$

$$\phi = 0 \quad \text{at } x = 0 \text{ and } x = L$$

$$\phi = 0 \quad \text{at } y = 0 \text{ and } y = H$$

where ϕ is the stream function.

$$\Delta \phi = 0$$

$$\frac{\partial \phi}{\partial x} = 0 \quad \text{at } x = 0 \text{ and } x = L$$

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{at } y = 0 \text{ and } y = H$$

$$\phi = 0 \quad \text{at } x = 0 \text{ and } x = L$$

$$\phi = 0 \quad \text{at } y = 0 \text{ and } y = H$$

$$\phi = 0 \quad \text{at } x = 0 \text{ and } x = L$$

$$\phi = 0 \quad \text{at } y = 0 \text{ and } y = H$$

$$\phi = 0 \quad \text{at } x = 0 \text{ and } x = L$$

$$\phi = 0 \quad \text{at } y = 0 \text{ and } y = H$$

$$+ \frac{R^2(X_{1p})}{2} [\theta_{1p} + \sin \theta_{1p} \cos \theta_{1p}] + \frac{\bar{y}(X_{1p})S}{2} - R(\bar{y})(X_{1p})\left(\frac{X_{1p}}{R}\right)$$

$$EI_c \Delta u = \frac{(R-r)^3}{3} - \frac{(\bar{y})S^2}{8} + \frac{(\bar{y})(X_{1p})^2}{2} - (X_{1p}) \left[\frac{\bar{L}R + S(R-r)}{4} \right] + \frac{(\bar{y})(X_{1p})S}{2} - (\bar{y})(X_{1p})^2 \\ + \frac{R^3}{3} \cos^3 \theta_{1p} + \frac{R^2(X_{1p})}{2} [\theta_{1p} + \sin \theta_{1p} \cos \theta_{1p}]$$

Since $\bar{y} = \frac{\bar{L}R + S(R-r)}{2S}$ we write the above as

$$-\frac{(R-r)^3}{3} - \frac{(\bar{y})S^2}{8} + \frac{(\bar{y})(X_{1p})^2}{2} - \frac{(X_{1p})(\bar{y})(2S)}{4} + \frac{(\bar{y})(X_{1p})S}{2} - (\bar{y})(X_{1p})^2 \\ + \frac{R^3}{3} \cos^3 \theta_{1p} + \frac{R^2(X_{1p})}{2} [\theta_{1p} + \sin \theta_{1p} \cos \theta_{1p}] \\ = -\frac{(R-r)^3}{3} - \bar{y} \left[\frac{S^2}{8} + \frac{(X_{1p})^2}{2} \right] + \frac{R^3}{3} \cos^3 \theta_{1p} + \frac{R^2(X_{1p})}{2} [\theta_{1p} + \sin \theta_{1p} \cos \theta_{1p}]$$

(C) DETERMINATION OF ANGULAR DEFLECTION AT R.F.

$$\theta = \int_{\theta_{1p}}^A \frac{M ds}{EI}$$

$$EI_c \theta = \int_{\theta_{1p}}^A (R \sin \theta - X_{1p}) \frac{R d\theta}{\sec \theta} = \int_{\theta_{1p}}^A (R \sin \theta - X_{1p}) R \cos \theta d\theta \\ = \int_{\theta_{1p}}^A R^2 \sin \theta \cos \theta d\theta - \int_{\theta_{1p}}^A R(X_{1p}) \cos \theta d\theta \\ = \frac{R^2}{2} \left[\sin^2 \theta \right]_{\theta_{1p}}^A - R(X_{1p}) \left[\sin \theta \right]_{\theta_{1p}}^A \\ = \frac{R^2}{2} \left[\left(\frac{S}{2R} \right)^2 - \sin^2 \theta_{1p} \right] - R(X_{1p}) \left[\frac{S}{2R} - \sin \theta_{1p} \right] \\ = \frac{S^2}{8} - \frac{R^2}{2} \left(\frac{X_{1p}}{R} \right)^2 - \frac{(X_{1p})S}{2} + R(X_{1p}) \left(\frac{X_{1p}}{R} \right) \\ = \frac{S^2}{8} - \frac{(X_{1p})^2}{2} - \frac{(X_{1p})S}{2} + (X_{1p})^2 \\ = \frac{S^2}{8} + \frac{(X_{1p})^2}{2} - \frac{(X_{1p})S}{2}$$

(D) THE DETERMINATION OF THE REACTIONS AT R.P.

$$V_o = (-) \frac{\Delta V}{\delta v v} \quad H_o = (-) \frac{\Delta H}{\delta h h} \quad M_o = (-) \frac{\Delta M}{\delta \theta \theta}$$

$$V_o = (-) \frac{\frac{s^3}{24} + \frac{(x_{1p})^3}{6} - \frac{(x_{1p})s^2}{3}}{\frac{s^3}{12}}$$

$$H_o = \frac{\frac{(-)(R-r)^3}{3} - \bar{y} \left[\frac{s^2}{8} + \frac{(x_{1p})^2}{2} \right] + \frac{R^3 \cos^3 \theta_{1p}}{3} + \frac{R^2 (x_{1p})}{2} [\theta_{1p} + \sin \theta_{1p} \cos \theta_{1p}]}{(\bar{y})^2 s - R^2 (s) + \frac{s^3}{12}}$$

$$M_o = (-) \frac{\frac{s^2}{8} + \frac{(x_{1p})^2}{2} - \frac{(x_{1p})s}{2}}{s}$$

(E) THE DETERMINATION OF THE PLANE END MOMENTS

$$M_1^f = V_o \bar{y} - H_o [r - (R - \bar{y})] + M_o$$

$$\text{Note: } \bar{y} = \frac{LR + s(R-r)}{2s}$$

M_1^f is determined by writing moments about the right end.

A positive value of moment indicates tension on the top of the arch.

V. DETERMINATION OF RELATIONSHIP BETWEEN MOMENT AND SPAN FOR THE SAME RISE/SPAN RATIO.

Arches of the same Rise/Span ratio have been proved similar on Page 17. The moment areas of corresponding $\frac{Mds}{EI}$ loads have been proved to vary as $\frac{s^2}{s_1^2}$. These relationships hold for the non-prismatic arch as well as for the prismatic since the working line of each is an arc of a circle.

1. The following are the first three terms of an arithmetic progression (AP):

$$\frac{1}{b}, \frac{1}{b}, \frac{1}{b} \quad (1)$$

$$\frac{1}{b} + \frac{1}{b} + \frac{1}{b} = \frac{3}{b}$$

$$\left[\frac{1}{b} + \frac{1}{b} + \frac{1}{b} \right] \frac{1}{b} = \frac{3}{b^2}$$

$$\frac{3}{b^2} = \frac{3}{b^2}$$

2. The following are the first three terms of a geometric progression (GP):

$$1, 1, 1$$

3. The following are the first three terms of a harmonic progression (HP):

1, 1, 1

4. The following are the first three terms of a cubic progression:

1, 1, 1

From Page 14 It is seen that M at any point due to a 1 kip load on the Arch = $(R \sin \theta - R_{1p})$.

$$\text{For Arch 2, } M_2 = R_2 \sin \theta - R_2 \sin \theta_{1p} = R_2 (\sin \theta - \sin \theta_{1p})$$

$$\text{For Arch 1, } M_1 = R_1 \sin \theta - R_1 \sin \theta_{1p} = R_1 (\sin \theta - \sin \theta_{1p})$$

$$\frac{M_2}{M_1} = \frac{R_2}{R_1} = \frac{S_2}{S_1} \quad \text{or} \quad M_2 = M_1 \left(\frac{S_2}{S_1} \right)$$

Note: The above M 's are moment intensities acting on areas $\frac{ds}{EI_c \sec \theta}$, or the moment existing at the corresponding section of the cut back structure.

For Arch 1

$$\left[\frac{ds}{EI_c \sec \theta} \right]_1$$

For Arch 2

$$\left[\frac{ds}{EI_c \sec \theta} \right]_1 \times \frac{R_2}{R_1} = \left[\frac{ds}{EI_c \sec \theta} \right]_1 \times \frac{S_2}{S_1}$$

Since the moment intensities vary as $\frac{S_2}{S_1}$, and the areas on which these intensities act vary as $\frac{S_2}{S_1}$, the Elastic Loads $\left(\frac{M ds}{EI_c \sec \theta} \right)$ vary as $\left(\frac{S_2}{S_1} \right)^2$. The vertical and horizontal deflections at the N.F. caused by a load at any point on the arch consequently will vary as $\left(\frac{S_2}{S_1} \right)^3$.

Since the vertical and horizontal reactions at the N.P. of any arch are equal to the quotient of linear deflections, then the V_0 and H_0 will be independent of span. For a more rigid proof of this, see Page 18 .

For Arch 2

$$M_{02} = - \frac{\theta_2}{\alpha_2} = \frac{\int_0^A \left[\frac{Mds}{EI_c \sec \theta} \right]_2}{s_2} = \frac{\int_0^A \left[\frac{Mds}{EI_c \sec \theta} \right]_1}{s_1 \times \frac{s_2}{s_1}} \times \left(\frac{s_2}{s_1} \right)^2$$

For Arch 1

$$M_{01} = - \frac{\theta_1}{\alpha_1} = \frac{\int_0^A \left[\frac{Mds}{EI_c \sec \theta} \right]_1}{s_1}$$

$$\frac{M_{02}}{M_{01}} = \left(\frac{s_2}{s_1} \right)$$

$$M_{02} = M_{01} \frac{s_2}{s_1} \quad (M_0's \text{ vary directly as span})$$

Having proved that the vertical and horizontal reactions at the N.P. are independent of span, and that the moment reaction is directly proportional to span, we find by taking moments at the left end of the arch that the fixed end moment varies directly as the span. This proof is identical with that found on Page 19 .

It was found that the reaction of the system was not affected by the addition of a small amount of water. The reaction was found to be first order with respect to the concentration of the reactants. The rate of reaction was found to be independent of the concentration of the products.

$$\frac{d[A]}{dt} = -k[A] \quad \text{or} \quad \ln \frac{[A]_0}{[A]} = kt$$

$$\frac{d[B]}{dt} = k[A] \quad \text{or} \quad \ln \frac{[B]}{[A]_0} = kt$$

$$\frac{d[C]}{dt} = k[A] \quad \text{or} \quad \ln \frac{[C]}{[A]_0} = kt$$

$$k = \frac{1}{t} \ln \frac{[A]_0}{[A]} \quad \text{or} \quad k = \frac{1}{t} \ln \frac{[B]}{[A]_0}$$

The reaction was found to be first order with respect to the concentration of the reactants. The rate of reaction was found to be independent of the concentration of the products. The reaction was found to be first order with respect to the concentration of the reactants. The rate of reaction was found to be independent of the concentration of the products.

VI. DETERMINATION OF CARRY-OVER FACTOR

Cutting the arch at the left springing and applying redundants M_a , V_a and H_a at that point, the deflections at the left end will be as follows:

$$(1) \Delta V = M_a \delta'_{vm} + V_a \delta_{vv} + H_a \delta_{vh}$$

$$(2) \Delta H = M_a \delta'_{hm} + V_a \delta_{hv} + H_a \delta_{hh}$$

$$(3) \theta = M_a \alpha_{\theta m} + V_a \alpha'_{\theta v} + H_a \alpha'_{\theta h}$$

By multiplying both sides of equations (1), (2) and (3) by EI_c , these equations take the form:

$$(4) EI_c \Delta V = M_a (EI_c \delta'_{vm}) + V_a (EI_c \delta_{vv}) + H_a (EI_c \delta_{vh})$$

$$(5) EI_c \Delta H = M_a (EI_c \delta'_{hm}) + V_a (EI_c \delta_{hv}) + H_a (EI_c \delta_{hh})$$

$$(6) EI_c \theta = M_a (EI_c \alpha_{\theta m}) + V_a (EI_c \alpha'_{\theta v}) + H_a (EI_c \alpha'_{\theta h})$$

From the theories of virtual work and Maxwell's Law of Reciprocal Deflections:

(Refer to Page 21 for x and y values)

$$\begin{aligned} EI \delta'_{vm} &= EI \alpha'_{\theta v} = \int_{-A}^A \frac{x ds}{\sec \theta} = \int_{-A}^A \left(\frac{x}{2} + R \sin \theta \right) \cos \theta ds \\ &= \frac{SR}{2} \int_{-A}^A \cos \theta ds + R^2 \int_{-A}^A \sin \theta \cos \theta ds \\ &= \frac{SR}{2} \left[\sin \theta \right]_{-A}^A + \frac{R^2}{2} \left[\sin^2 \theta \right]_{-A}^A \\ &= \frac{SR}{2} \left[\frac{3}{2R} - (-) \frac{3}{2R} \right] + \frac{R^2}{2} [0] \\ &= \frac{SR}{2} \left(\frac{3}{R} \right) = \frac{5}{2} (S) \end{aligned}$$

Let us assume that the function $f(x)$ is continuous on the interval $[a, b]$ and that the function $g(x)$ is continuous on the interval $[c, d]$. Then the function $h(x)$ is continuous on the interval $[a, b]$.

$$f(x) = x^2, \quad g(x) = x^3, \quad h(x) = x^4 \quad (1)$$

$$f(x) = x^2, \quad g(x) = x^3, \quad h(x) = x^4 \quad (2)$$

$$f(x) = x^2, \quad g(x) = x^3, \quad h(x) = x^4 \quad (3)$$

Let us assume that the function $f(x)$ is continuous on the interval $[a, b]$ and that the function $g(x)$ is continuous on the interval $[c, d]$. Then the function $h(x)$ is continuous on the interval $[a, b]$.

$$f(x) = x^2, \quad g(x) = x^3, \quad h(x) = x^4 \quad (4)$$

$$f(x) = x^2, \quad g(x) = x^3, \quad h(x) = x^4 \quad (5)$$

$$f(x) = x^2, \quad g(x) = x^3, \quad h(x) = x^4 \quad (6)$$

Let us assume that the function $f(x)$ is continuous on the interval $[a, b]$ and that the function $g(x)$ is continuous on the interval $[c, d]$. Then the function $h(x)$ is continuous on the interval $[a, b]$.

$$f(x) = x^2, \quad g(x) = x^3, \quad h(x) = x^4 \quad (7)$$

$$f(x) = x^2, \quad g(x) = x^3, \quad h(x) = x^4 \quad (8)$$

$$f(x) = x^2, \quad g(x) = x^3, \quad h(x) = x^4 \quad (9)$$

$$f(x) = x^2, \quad g(x) = x^3, \quad h(x) = x^4 \quad (10)$$

$$f(x) = x^2, \quad g(x) = x^3, \quad h(x) = x^4 \quad (11)$$

$$\begin{aligned}
 EI \delta'_{nu} &= EI \alpha'_{en} = \int_{-A}^A \frac{y ds}{\sec \theta} = \int_{-A}^A [R \cos \theta - (R-r)] \cos \theta R d\theta \\
 &= 2 \int_0^A R^2 \cos^2 \theta d\theta - 2(R-r) \int_0^A \cos \theta d\theta \\
 &= 2R^2 \left[\frac{\theta}{2} + \frac{\sin \theta \cos \theta}{2} \right]_0^A - 2(R-r) R [\sin \theta]_0^A \\
 &= R^2 \left[\frac{1}{2R} + \frac{3}{2R} \frac{(R-r)}{R} \right] - 2(R-r) R \left(\frac{3}{2R} \right) \\
 &= \frac{1}{2} [1R + 3(R-r)] - 3(R-r) \\
 &= \left[\frac{1R + 3(R-r)}{2R} - (R-r) \right] S
 \end{aligned}$$

$$EI \alpha_{en} = \int_{-A}^A \frac{ds}{\sec \theta} = \int_0^A R \cos \theta d\theta = 2R [\sin \theta]_0^A = 2R \left(\frac{3}{2R} \right) = 3$$

$$\begin{aligned}
 EI \delta'_{va} &= EI \delta'_{hv} = \int_{-A}^A xy \frac{ds}{\sec \theta} \\
 &= \int_{-A}^A \left[\frac{3}{2} + s \sin \theta \right] [R \cos \theta - (R-r)] \cos \theta R d\theta \\
 &= \frac{3R^2}{2} \int_{-A}^A \cos^2 \theta d\theta + R^3 \int_{-A}^A \cos^2 \theta \sin \theta d\theta - \frac{3}{2}(R-r) R \int_{-A}^A \cos \theta d\theta - R^2(R-r) \int_{-A}^A \sin \theta \cos \theta d\theta \\
 &= \frac{3R^2}{2} \left[\frac{\theta}{2} + \frac{\sin \theta \cos \theta}{2} \right]_{-A}^A - \frac{R^3}{3} [\cos^3 \theta]_{-A}^A - \frac{3}{2}(R-r) R [\sin \theta]_{-A}^A - \frac{R^2(R-r)}{2} [\sin^2 \theta]_{-A}^A \\
 &= \frac{3}{2} R^2 \left[\frac{1}{2R} + \frac{3}{2R} \frac{(R-r)}{R} \right] - 0 - \frac{3}{2} (R-r) R \left(\frac{3}{R} \right) - 0 \\
 &= \frac{3}{2} \left[\frac{1R + 3(R-r)}{2R} \right] - \frac{3}{2} (R-r) \\
 &= \left(\frac{3}{2} \right) \left[\frac{1R + 3(R-r)}{2R} - (R-r) \right] (S)
 \end{aligned}$$

$$\lim_{n \rightarrow \infty} \left[\frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) \right] = \int_0^1 f(x) dx$$

$$\lim_{n \rightarrow \infty} \left[\frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) \right] = \int_0^1 f(x) dx$$

$$\left[\frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) \right] = \int_0^1 f(x) dx$$

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$$\lim_{n \rightarrow \infty} \left[\frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) \right] = \int_0^1 f(x) dx$$

$$\lim_{n \rightarrow \infty} \left[\frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) \right] = \int_0^1 f(x) dx$$

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$$\left[\frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) \right] = \int_0^1 f(x) dx$$

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$$\left[\frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) \right] = \int_0^1 f(x) dx$$

$$\left[\frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) \right] = \int_0^1 f(x) dx$$

$$\begin{aligned}
 MI_{\delta vv} &= \int_{-A}^A x^2 \frac{ds}{\sec \theta} = \int_{-A}^A \left(\frac{R}{2} + R \sin \theta \right)^2 \cos \theta R d\theta \\
 &= \int_{-A}^A \left(\frac{R^2}{4} + 2R^2 \sin \theta + R^2 \sin^2 \theta \right) \cos \theta d\theta \\
 &= \frac{R^2}{4} \int_{-A}^A \cos \theta d\theta + 2R^2 \int_{-A}^A \sin \theta \cos \theta d\theta + R^2 \int_{-A}^A \sin^2 \theta \cos \theta d\theta \\
 &= \frac{R^2}{4} \left[\sin \theta \right]_{-A}^A + \frac{R^2}{2} \left[\sin^2 \theta \right]_{-A}^A + R^2 \left[\frac{\sin^3 \theta}{3} \right]_{-A}^A \\
 &= \frac{R^2}{4} (A) \left(\frac{R}{R} \right) + \frac{R^2}{2} [0] + \frac{R^3}{3} \left[\left(\frac{R}{2R} \right)^3 + \left(\frac{R}{2R} \right)^3 \right] \\
 &= \frac{R^3}{4} + \frac{2R^3}{3} \frac{R^3}{8R^3} \\
 &= \frac{R^3}{4} + \frac{R^3}{12} = \frac{R^3}{3}
 \end{aligned}$$

$$\begin{aligned}
 MI_{\delta hh} &= \int_{-A}^A y^2 \frac{ds}{EI} = 2 \int_0^A [R \cos \theta - (R-r)]^2 \cos \theta R d\theta \\
 &= 2 \int_0^A R^3 \cos^3 \theta - 2 \int_0^A 2R^2 (R-r) \cos^2 \theta d\theta + 2 \int_0^A R(R-r)^2 \cos \theta d\theta \\
 &= 2R^3 \left[\sin \theta - \frac{\sin^3 \theta}{3} \right]_0^A - 4R^2 (R-r) \left[\frac{\theta}{2} + \frac{\sin \theta \cos \theta}{2} \right]_0^A + 2R(R-r)^2 \left[\sin \theta \right]_0^A \\
 &= 2R^3 \left(\frac{R}{2R} \right) - \frac{2R^3}{3} \left[\frac{R^3}{8R^3} \right] - \frac{4R^2 (R-r)}{2} \left[\frac{\widehat{L}R}{2R} + \frac{R}{2R} \frac{(R-r)}{R} \right] + 2R(R-r)^2 \left(\frac{R}{2R} \right) \\
 &= 2R^2 - \frac{R^3}{12} - \frac{4R^2 (R-r)}{2} \left[\frac{\widehat{L}R + R(R-r)}{2R^2} \right] + R(R-r)^2 \\
 &= 2R^2 - \frac{R^3}{12} - (R-r) \left[\widehat{L}R + R(R-r) \right] + R(R-r)^2 \\
 &= 2R^2 - \frac{R^3}{12} - \widehat{L}R(R-r)
 \end{aligned}$$

$$\text{momentum } (p_1, p_2) = \frac{1}{2} \sum_{i=1}^n \frac{p_i^2}{m_i} \sum_{j=1}^n \frac{1}{m_j}$$

$$\text{momentum } (p_1, p_2) = \frac{1}{2} \sum_{i=1}^n \frac{p_i^2}{m_i} + \frac{1}{2} \sum_{j=1}^n \frac{1}{m_j}$$

$$\text{momentum } (p_1, p_2) = \frac{1}{2} \sum_{i=1}^n \frac{p_i^2}{m_i} + \frac{1}{2} \sum_{j=1}^n \frac{1}{m_j} =$$

$$\left[\frac{p_1^2}{2m_1} \right] + \left[\frac{p_2^2}{2m_2} \right] + \left[\frac{1}{2m_1} \right] + \left[\frac{1}{2m_2} \right] =$$

$$\left[\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \right] + \left[\frac{1}{2m_1} + \frac{1}{2m_2} \right] =$$

$$\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2m_1} + \frac{1}{2m_2} =$$

$$\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2m_1} + \frac{1}{2m_2} =$$

$$\text{momentum } (p_1, p_2) = \frac{1}{2} \sum_{i=1}^n \frac{p_i^2}{m_i} + \frac{1}{2} \sum_{j=1}^n \frac{1}{m_j}$$

$$\text{momentum } (p_1, p_2) = \frac{1}{2} \sum_{i=1}^n \frac{p_i^2}{m_i} + \frac{1}{2} \sum_{j=1}^n \frac{1}{m_j} =$$

$$\left[\frac{p_1^2}{2m_1} \right] + \left[\frac{p_2^2}{2m_2} \right] + \left[\frac{1}{2m_1} \right] + \left[\frac{1}{2m_2} \right] =$$

$$\left[\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \right] + \left[\frac{1}{2m_1} + \frac{1}{2m_2} \right] =$$

$$\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2m_1} + \frac{1}{2m_2} =$$

$$\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2m_1} + \frac{1}{2m_2} =$$

$$\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2m_1} + \frac{1}{2m_2} =$$

To obtain the carry-over factor we hinge the arch at the left end, set equations (4) and (5) equal to zero and solve simultaneously for V_a and H_a in terms of M_a . Then by taking moments at the fixed end, M_b can be expressed as the carry-over factor times M_a .

The resulting equations for V_a and H_a

$$V_a = D_2' M_a$$

$$H_a = D_1' M_a$$

will be the same as those for the prismatic arch found on Page 22. The numerical values of D_2 and D_1 will be different due to the difference in the values of the deflections. Hence for the non-prismatic arch, we write

$$V_a = D_2' M_a$$

$$H_a = D_1' M_a$$

taking moments at the fixed end

$$M_b = M_a + V_a \bar{x}$$

$$= M_a + (D_2') M_a \bar{x} = (1 + D_2' \bar{x}) M_a$$

$$\therefore \text{The Carry-Over Factor} = (-)(1 + D_2' \bar{x})$$

VI.(a) DETERMINATION OF RELATIONSHIP BETWEEN CARRY-OVER FACTOR AND SPAN FOR THE SAME RISE/SPAN RATIO.

Prismatic Arches of any span length, having the same rise/span ratio have been proved to be similar. The non-prismatic arches of the same rise/span ratio are also similar since $\sec \theta$ is the same for the corresponding points on the arches. Since the similarity holds for

the first part, the experiment (1) and (2) were carried out
 under identical conditions for $\frac{1}{2}$ and $\frac{1}{4}$ of the total time.
 At the end of the experiment the first part was found to be
 the same as the second part.

The following relations hold for $\frac{1}{2}$ and $\frac{1}{4}$

$$p^2 = \frac{1}{2} p^2$$

$$p^2 = \frac{1}{4} p^2$$

It is seen from these relations that the first part is
 the same as the second part. The first part is
 the same as the second part. The first part is
 the same as the second part.

$$p^2 = \frac{1}{2} p^2$$

$$p^2 = \frac{1}{4} p^2$$

The following relations hold for $\frac{1}{2}$ and $\frac{1}{4}$

$$p^2 = \frac{1}{2} p^2$$

$$p^2 = \frac{1}{4} p^2$$

$$p^2 = \frac{1}{2} p^2$$

The following relations hold for $\frac{1}{2}$ and $\frac{1}{4}$

The following relations hold for $\frac{1}{2}$ and $\frac{1}{4}$

non-prismatic arches the deflections for non-prismatic arches will vary with span in the same ratio as do the deflections for the prismatic arch (See Page 24).

For the Non-Prismatic Arch

$$V_a = D_2' \left(\frac{S_1}{S_2} \right) M_a$$

$$M_a = D_1' \left(\frac{S_1}{S_2} \right) M_a$$

Taking moments at the fixed end of Arch 2 of span $S_2 =$

$$S_1 \left(\frac{S_2}{S_1} \right)$$

$$M_b = M_a + V_a S_2 = M_a + D_2' \left(\frac{S_1}{S_2} \right) M_a S_2$$

$$= M_a + D_2' \left(\frac{S_1}{S_2} \right) M_a S_1 \left(\frac{S_2}{S_1} \right)$$

$$= M_a + D_2' (S_1) M_a$$

$$= (1 + D_2' S_1) M_a$$

The Carry-Over Factor for Arch 2 = $(-)(1 + D_2' S_1)$ which is the same as the C.O.F. for Arch.1.

\therefore Carry-Over Factor for arches of same Rise/Span ratio is independent of span.

VII. DETERMINATION OF ABSOLUTE STIFFNESS

The equation for the absolute stiffness of the non-prismatic arch would take the same form as the equation on Page 25 for the prismatic arch. The numerical value of D_3 would be different and the I for the non-prismatic would

...
...
...
...
...

$$x^2 \left(\frac{1}{x^2} \right) = x^0 = 1$$

$$x^2 \left(\frac{1}{x^2} \right) = x^0 = 1$$

...
...

$$x^2 \left(\frac{1}{x^2} \right) = x^0 = 1$$

$$\left(\frac{1}{x^2} \right) x^2 = x^0 = 1$$

$$x^2 \left(\frac{1}{x^2} \right) = x^0 = 1$$

$$x^2 \left(\frac{1}{x^2} \right) = x^0 = 1$$

...
...

...
...

...

...
...
...
...

be expressed as the I at the crown, I_c .

Hence for the absolute stiffness of the non-prismatic arch

$$K_A = \frac{EI_c}{L_3'} = \text{Absolute stiffness}$$

VII(a) DETERMINATION OF RELATIONSHIP BETWEEN ABSOLUTE STIFFNESS AND SPAN FOR THE SAME RISE/SPAN RATIO

Since the deflections for the non-prismatic arch vary with span in the same ratio as do the deflections for the prismatic arch, the equation for the absolute stiffness for Arch 2 written in terms of the absolute stiffness for Arch 1 is in form the same as the equation on Page 26 .

$$K_A = \frac{EI_c}{L_3'} \cdot \frac{S_1}{S_2}$$

∴ Absolute Stiffness varies inversely as the span.

VIII. DETERMINATION OF THE INDUCED MOMENT AND REACTIONS (VERTICAL AND HORIZONTAL) OCCURRING AT THE END OF AN ARCH DUE TO SPREADING OF THE ARCH UNDER LOAD.

The equation for V_A and H_A in terms of K_A , for the non-prismatic arch take the same form as those equations found on Page 22 .

By substituting the values of the deflections in the numerator of the equation for V_A we prove that $V_A = 0$ for the non-prismatic arch also.

$$\frac{S}{2}(S) \left[\frac{LH + S(R-r)}{2S} - (R-r) \right] S - S\left(\frac{S}{2}\right) \left[\frac{LH + S(R-r)}{2S} - (R-r) \right] S = 0$$

be considered as a function of x and y and the partial derivatives of u with respect to x and y are

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \left(\frac{1}{2} x^2 + y^2 \right) = x$$

(*) The partial derivatives of u with respect to x and y are $\frac{\partial u}{\partial x} = x$ and $\frac{\partial u}{\partial y} = 2y$.

Thus the partial derivatives of u are $\frac{\partial u}{\partial x} = x$ and $\frac{\partial u}{\partial y} = 2y$. The partial derivatives of u with respect to x and y are $\frac{\partial u}{\partial x} = x$ and $\frac{\partial u}{\partial y} = 2y$. The partial derivatives of u with respect to x and y are $\frac{\partial u}{\partial x} = x$ and $\frac{\partial u}{\partial y} = 2y$. The partial derivatives of u with respect to x and y are $\frac{\partial u}{\partial x} = x$ and $\frac{\partial u}{\partial y} = 2y$.

$$\frac{\partial u}{\partial x} = x, \quad \frac{\partial u}{\partial y} = 2y$$

Thus the partial derivatives of u are $\frac{\partial u}{\partial x} = x$ and $\frac{\partial u}{\partial y} = 2y$.

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Thus the partial derivatives of u are $\frac{\partial u}{\partial x} = x$ and $\frac{\partial u}{\partial y} = 2y$.

$$\frac{\partial u}{\partial x} = x, \quad \frac{\partial u}{\partial y} = 2y$$

Since $V_a = 0$, the equations for M_a , and H_a in terms of spread Δ_H will take the same form as the equations on Pages 272E.

Modified for the non-prismatic arch, the equations are:

$$M_a = \frac{EI_c \Delta_H}{D_4'}$$

$$H_a = \frac{EI_c \Delta_H}{D_5'}$$

VIII(a). DETERMINATION OF THE RELATIONSHIP BETWEEN THE REACTIONS, INDUCED AS A RESULT OF SPREAD, AND THE SPAN FOR THE SAME RISE/SPAN RATIO.

Since the deflections for the non-prismatic arch vary with span in the same ratio as do the deflections for the prismatic arch, the equations for M_a and H_a for Arch 2 written in terms of their values for Arch 1 are

$$M_a = \frac{EI_c \Delta_H}{D_4'} \left(\frac{S_1}{S_2}\right)^2$$

$$H_a = \frac{EI_c \Delta_H}{D_5'} \left(\frac{S_1}{S_2}\right)^3$$

$\therefore M_a$ varies inversely as the square of the span.

$\therefore H_a$ varies inversely as the cube of the span.

RESULTS

CONCLUSIONS

In this thesis, we have presented data in both curve and tabular form that will enable an engineer to solve any circular arched bent, the arched member of which

- (a) has a rise/span ratio between 0.04 and 0.40,
- (b) is of constant moment of inertia or,
- (c) is of varying moment of inertia, provided the moment of inertia at any point is equal to the "I" at the crown times the secant of the angle formed by a tangent to the center line of the arch at the point in question and a line parallel to the springing line.

"Fixed-end-moment" influence lines have been plotted by using the moments as caused by concentrated loads. By taking a summation of the ordinates of the influence lines, coefficients have been determined for formulas expressing the fixed-end-moment as caused by a complete uniform load; therefore, the arches may be solved for either concentrated or uniform loads.

Should the designer desire the required factors to a degree of accuracy greater than is available from the curves, exact formulas that will yield the factors are provided. A

complete solution for all factors can be obtained by knowing r (the rise in ft.), S (the span in ft.), and X_{1p} (the horizontal distance from the crown to the point of application of each load.) All formulas are based on loading to the right of the crown.

As a result of this investigation, the following relationships were proved:

For the prismatic arches having the same rise/span ratio:

- (1) The shear and thrust are constant for all spans.
- (2) The P.M.M. is directly proportional to span.
- (3) The carry-over factor is constant for all spans.
- (4) The absolute stiffness varies inversely as the span.

For the non-prismatic arches having the same rise/span ratio:

- (1) The shear and thrust are constant for all spans.
- (2) The P.M.M. is directly proportional to span.
- (3) The carry-over factor is constant for all spans.
- (4) The absolute stiffness varies inversely as the span.

For the non-prismatic arches regardless of rise/span ratio:

- (1) The shear is constant.

From the two problems illustrating the application of the method, it is evident that with the required factors available, the engineer can solve a circular arched bent in practically the same time that he would need in solving a rectangular bent.

Although we have limited our application to a single

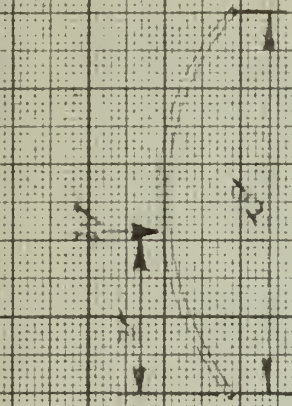
bent, the factors supplied in this thesis can be of particular advantage in extending the application of the moment distribution method to a series of continuous arches.

CURVES FOR PRISMATIC ARCH

ORIGINATING ARCH

BASE/SPAN

0.04
0.08
0.20
0.30
0.40



FOR CONC. LOAD $M = W \cdot L^2$
FOR UNIF. LOAD $M = w \cdot L^2$

VALUES OF $\frac{R}{S}$

0.04
0.20
0.30
0.40

FOR FINAL (LEFT)

ENTER $\frac{R}{S}$

FOR FINAL (RIGHT)

ENTER $\frac{R}{S}$

RISE/SPAN

0.04 0.00611
0.08 0.0077
0.20 0.00927
0.30 0.00951
0.40 0.00953

0.07

0.06

0.05

0.04

0.03

0.02

0.01

0.00

0.001

0.002

0.003

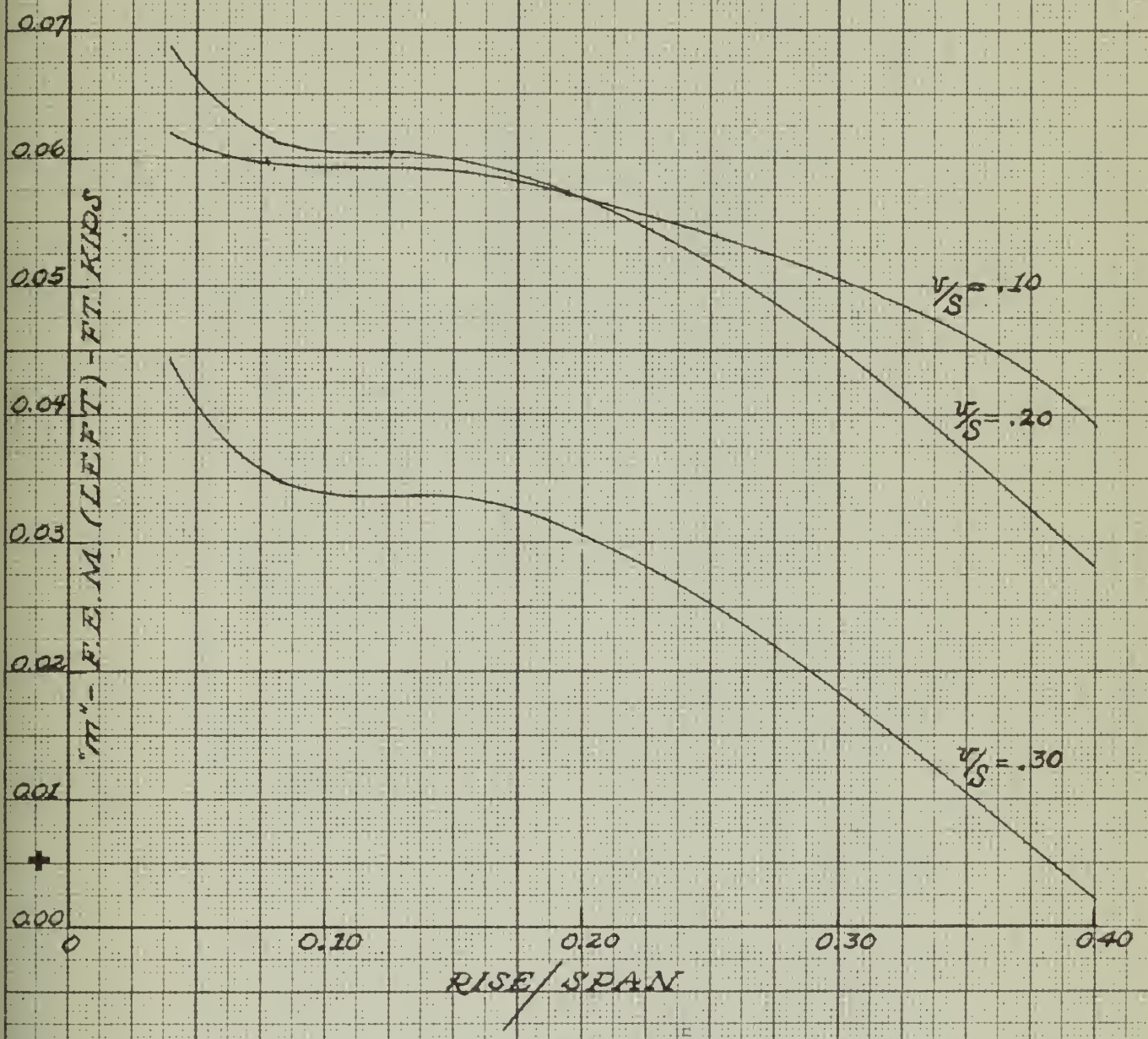
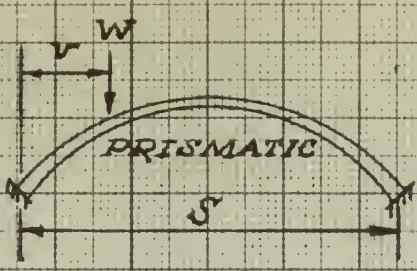
0.004

0.005

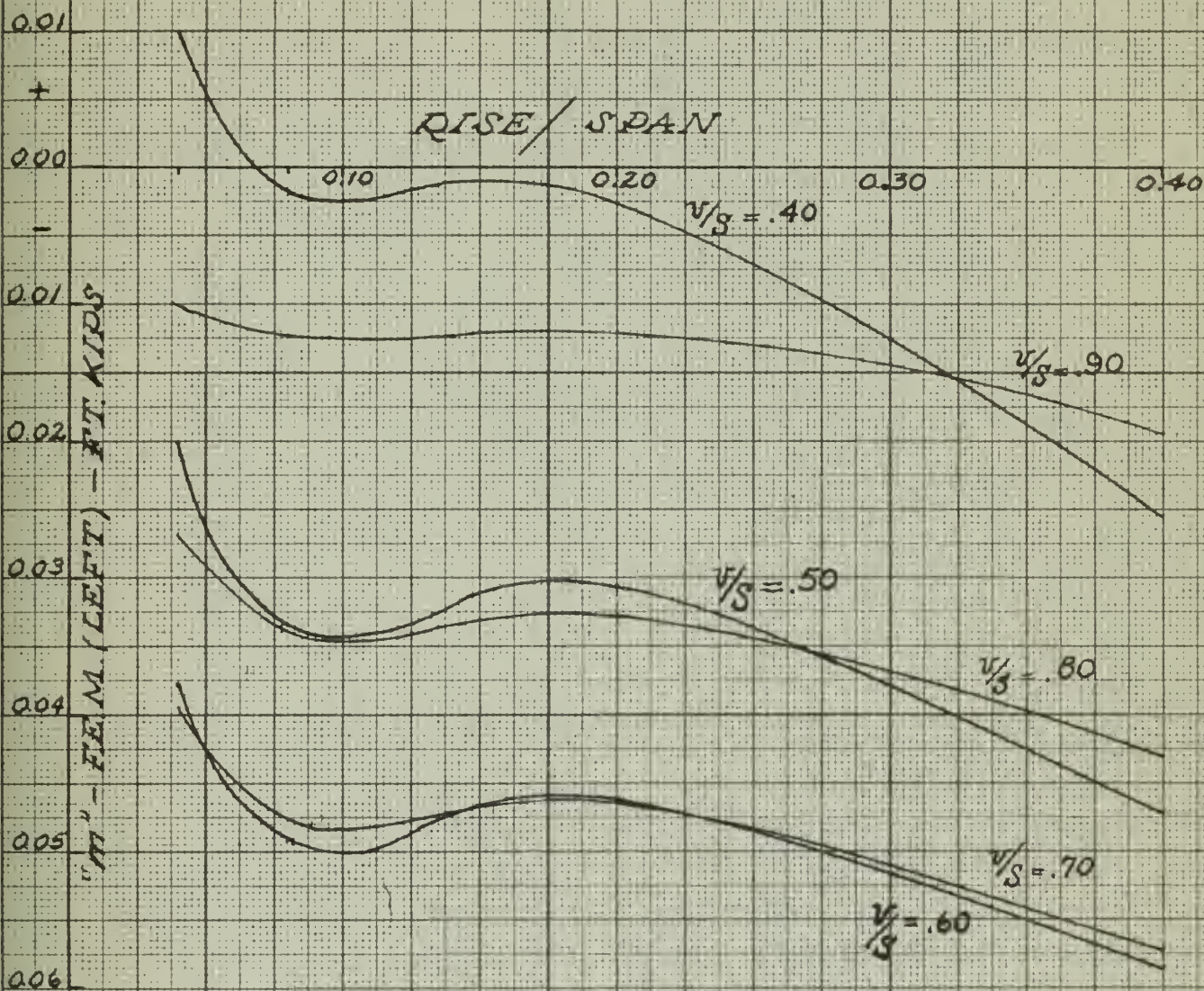
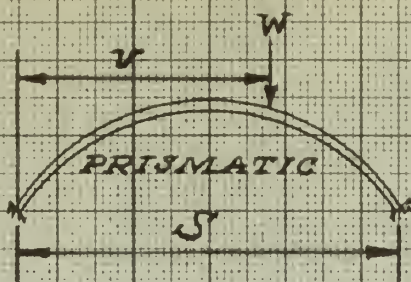
0.006

0.007

FIXED END MOMENT - P

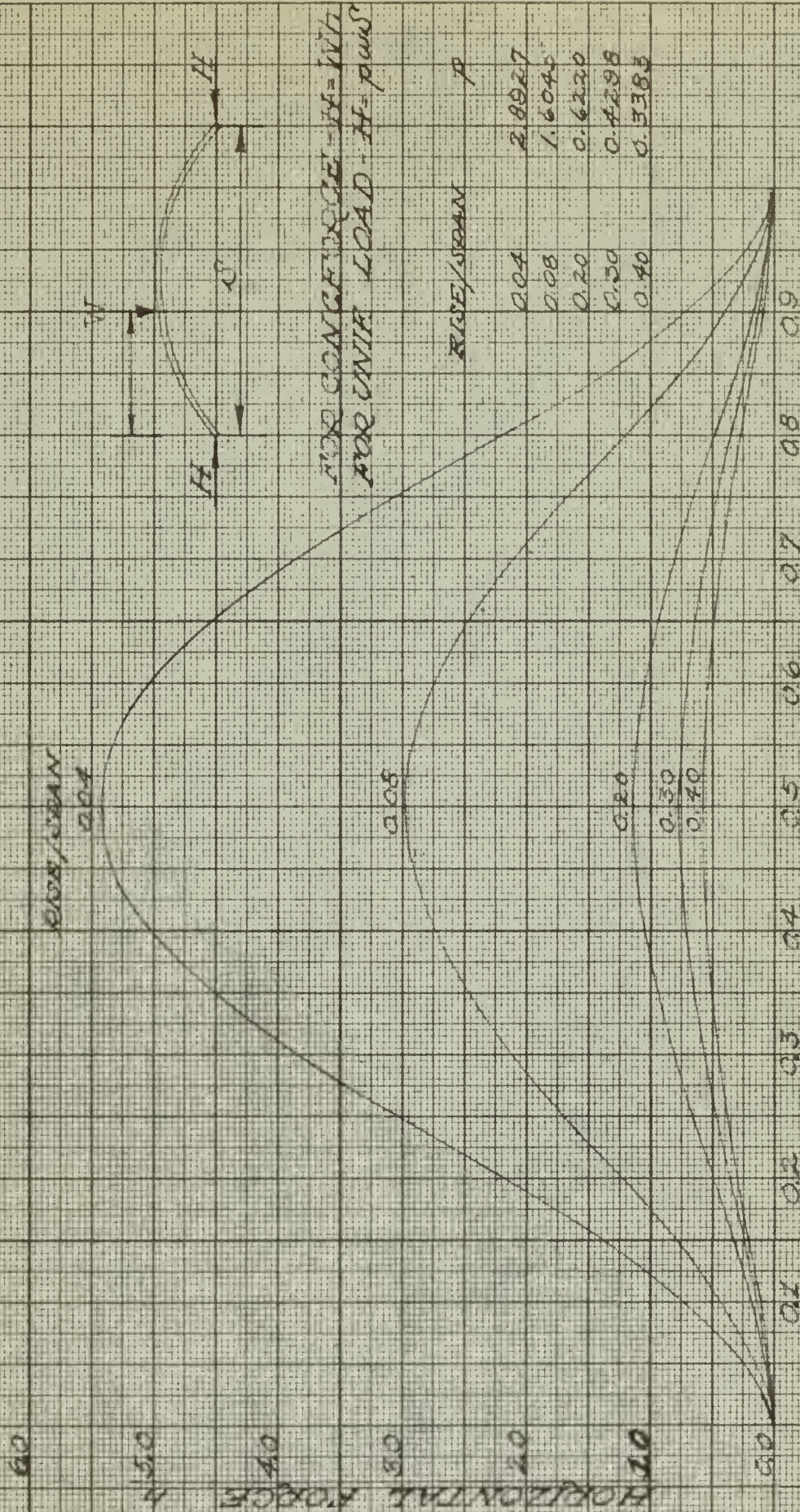


$$F.E.M. = WmS$$

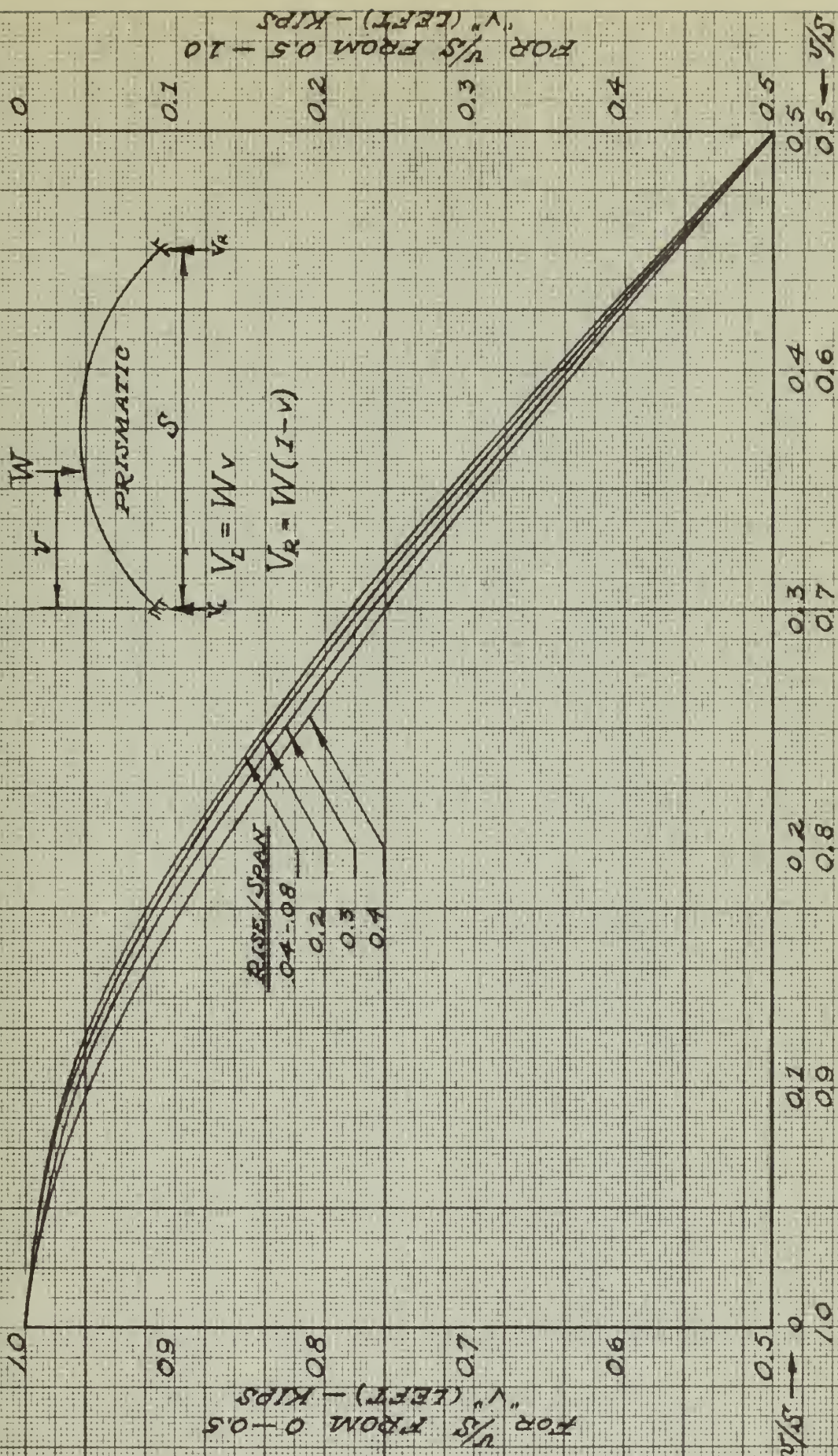


$$F.E.M. = W.M.S$$

PRISMATIC ARCH



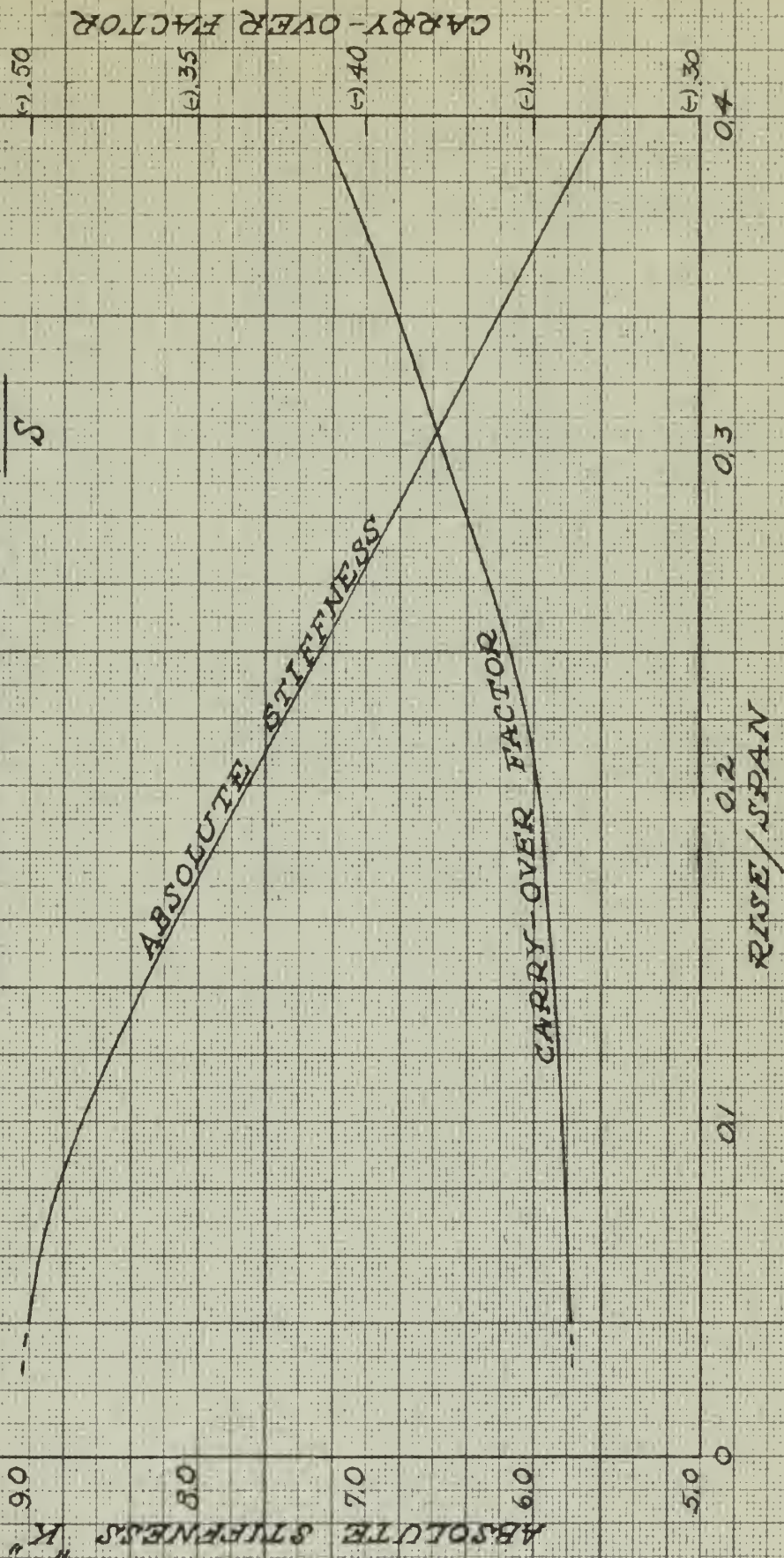
VALUES OF f

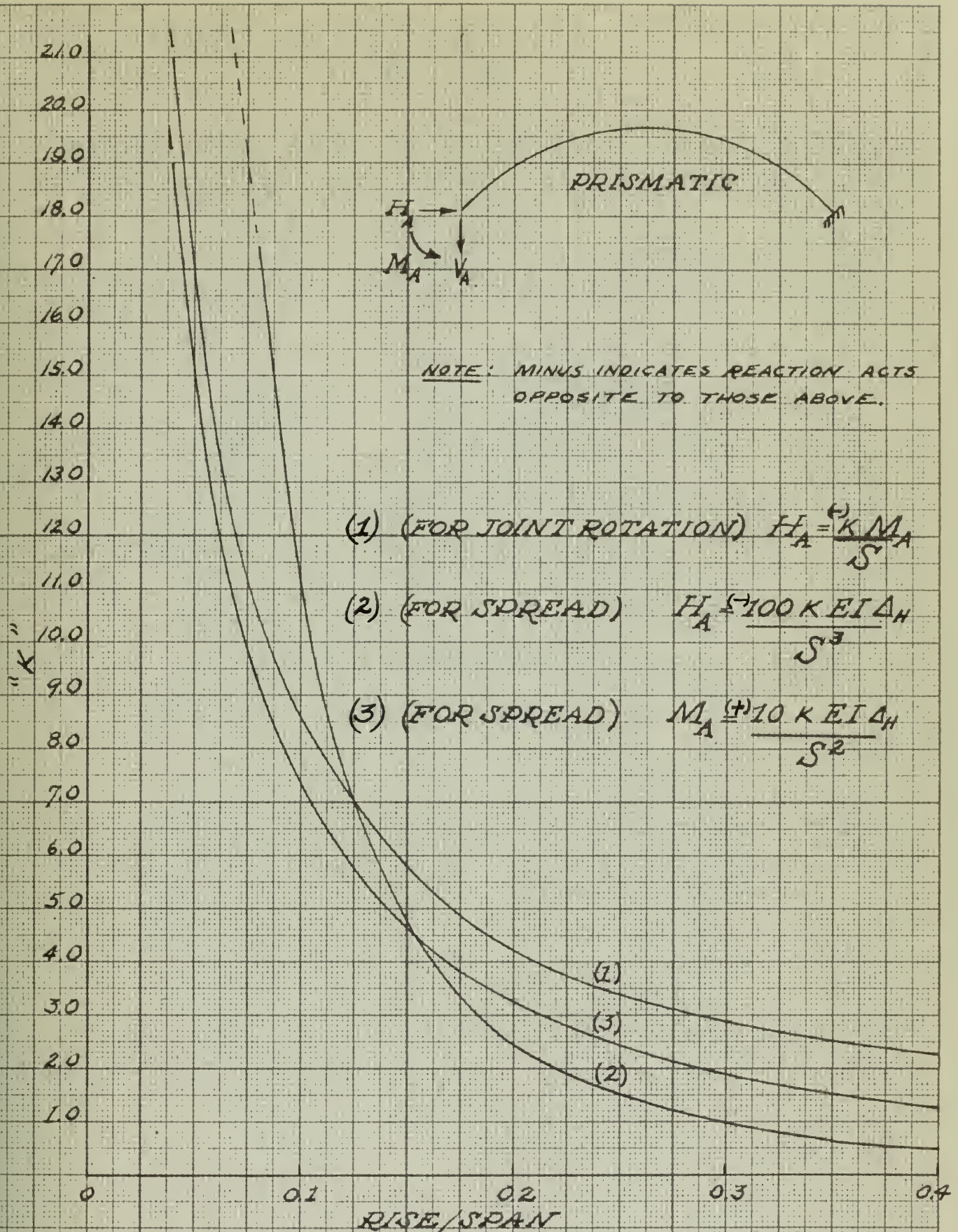


FOR PRISMATIC ARCH

CARRY-OVER FACTOR - READ CURVE

$$\text{ABSOLUTE STIFFNESS} = \frac{KEI}{S}$$





RESULTS FOR PNEUMATIC ARCH

BASED ON UNIT STAN

WOMANLY VIRTUES

WOMANLY VIRTUES

TABLE OF CONSTANTS FOR A PRISMATIC DAM
BASED ON A UNIT SPAN

Rise/Span = 0.04

Load Point	Pen (Left) Ft. LIPS	Pen (Right) Ft. LIPS	V (Left)	H (Thrust)
1	+.061961 S	-.009910 S	.971391	.710188
2	+.068696 S	-.028942 S	.935637	2.214017
3	+.044320 S	-.039502 S	.793922	3.841791
4	+.009969 S	-.037813 S	.647783	5.015055
5	-.020077 S	-.020077 S	.500000	5.432771

Rise/Span = 0.08

1	+.059115 S	-.012327 S	.971142	.403064
2	+.061345 S	-.033929 S	.935274	1.236693
3	+.034976 S	-.047651 S	.782626	2.077023
4	-.002109 S	-.049353 S	.647244	2.718642
5	-.033471 S	-.033471 S	.500000	2.936311

Rise/Span = 0.20

1	+.056678 S	-.012216 S	.968894	.165382
2	+.056894 S	-.032323 S	.938650	.495918
3	+.030630 S	-.046404 S	.777114	.820233
4	-.002575 S	-.046224 S	.643643	1.043354
5	-.030651 S	-.030651 S	.500000	1.130656

Rise/Span = 0.30

1	+.050830 S	-.014481 S	.963001	.125336
2	+.045027 S	-.038925 S	.932010	.355311
3	+.018267 S	-.061103 S	.789372	.567816
4	-.012708 S	-.061624 S	.638916	.711606
5	-.037923 S	-.037923 S	.500000	.761935

Rise/Span = 0.40

1	+.035990 S	-.019356 S	.956347	.117935
2	+.028206 S	-.043089 S	.971235	.293130
3	+.009170 S	-.057118 S	.759339	.444331
4	-.025480 S	-.058521 S	.633042	.643263
5	-.047193 S	-.047193 S	.500000	.577442

TABLE 1. LIST OF THE NAMES OF THE CITIES
IN THE YEAR 1917 AND 1918

TABLE 1. LIST OF THE NAMES OF THE CITIES

1917	1918	1917	1918	1917
1917	1918	1917	1918	1917

1917	1918	1917	1918	1917
1917	1918	1917	1918	1917
1917	1918	1917	1918	1917
1917	1918	1917	1918	1917
1917	1918	1917	1918	1917

TABLE 2. LIST OF THE NAMES OF THE CITIES

1917	1918	1917	1918	1917
1917	1918	1917	1918	1917
1917	1918	1917	1918	1917
1917	1918	1917	1918	1917
1917	1918	1917	1918	1917

TABLE 3. LIST OF THE NAMES OF THE CITIES

1917	1918	1917	1918	1917
1917	1918	1917	1918	1917
1917	1918	1917	1918	1917
1917	1918	1917	1918	1917
1917	1918	1917	1918	1917

TABLE 4. LIST OF THE NAMES OF THE CITIES

1917	1918	1917	1918	1917
1917	1918	1917	1918	1917
1917	1918	1917	1918	1917
1917	1918	1917	1918	1917
1917	1918	1917	1918	1917

TABLE 5. LIST OF THE NAMES OF THE CITIES

1917	1918	1917	1918	1917
1917	1918	1917	1918	1917
1917	1918	1917	1918	1917
1917	1918	1917	1918	1917
1917	1918	1917	1918	1917

Tabulation of Results for Prismatic Arch

<u>Rise/Span</u>	<u>Absolute Stiffness</u>	<u>Carry-over Factor</u>
0.04	9.009735 EI/S	(-) .3391362
0.08	8.834828 EI/S	(-) .3406277
0.20	7.695889 EI/S	(-) .3438530
0.30	6.631018 EI/S	(-) .3777210
0.40	5.591090 EI/S	(-) .4148421

With Joint A free to rotate, and a moment of M_a ft.kips applied at A, the following thrusts and shears are induced at A.

<u>Rise/Span</u>	<u>Thrust (KIPS)</u>	<u>Vertical Shear (KIPS)</u>
0.04	20.937421 M_a/S	.6606638 M_a/S
0.08	10.498726 M_a/S	.6593723 M_a/S
0.20	4.216799 M_a/S	.6511470 M_a/S
0.30	2.839591 M_a/S	.6222790 M_a/S
0.40	2.240643 M_a/S	.5851579 M_a/S

When a spread of ΔH feet occurs, the following moments, shears and thrusts occur.

<u>Rise/Span</u>	<u>Thrust (KIPS)</u>	<u>Moment (Ft.KIPS)</u>	<u>Shear</u>
0.04	7064.9449 EI $\Delta H/S^3$	(-) 188.64038 EI $\Delta H/S^2$	0
0.08	1740.5620 EI $\Delta H/S^3$	(-) 92.67798 EI $\Delta H/S^2$	0
0.20	245.8247 EI $\Delta H/S^3$	(-) 32.45204 EI $\Delta H/S^2$	0
0.30	97.8593 EI $\Delta H/S^3$	(-) 19.16094 EI $\Delta H/S^2$	0
0.40	48.5544 EI $\Delta H/S^3$	(-) 12.52766 EI $\Delta H/S^2$	0

TABLE 1. SUMMARY OF DATA FOR THE FIRST SET OF EXPERIMENTS

Run	Time (min)	Temperature (°C)	Pressure (mm Hg)
1	10.0	100.0	760.0
2	20.0	100.0	760.0
3	30.0	100.0	760.0
4	40.0	100.0	760.0
5	50.0	100.0	760.0

These data were obtained from the first set of experiments. The values are given in the units indicated. The values are given in the units indicated.

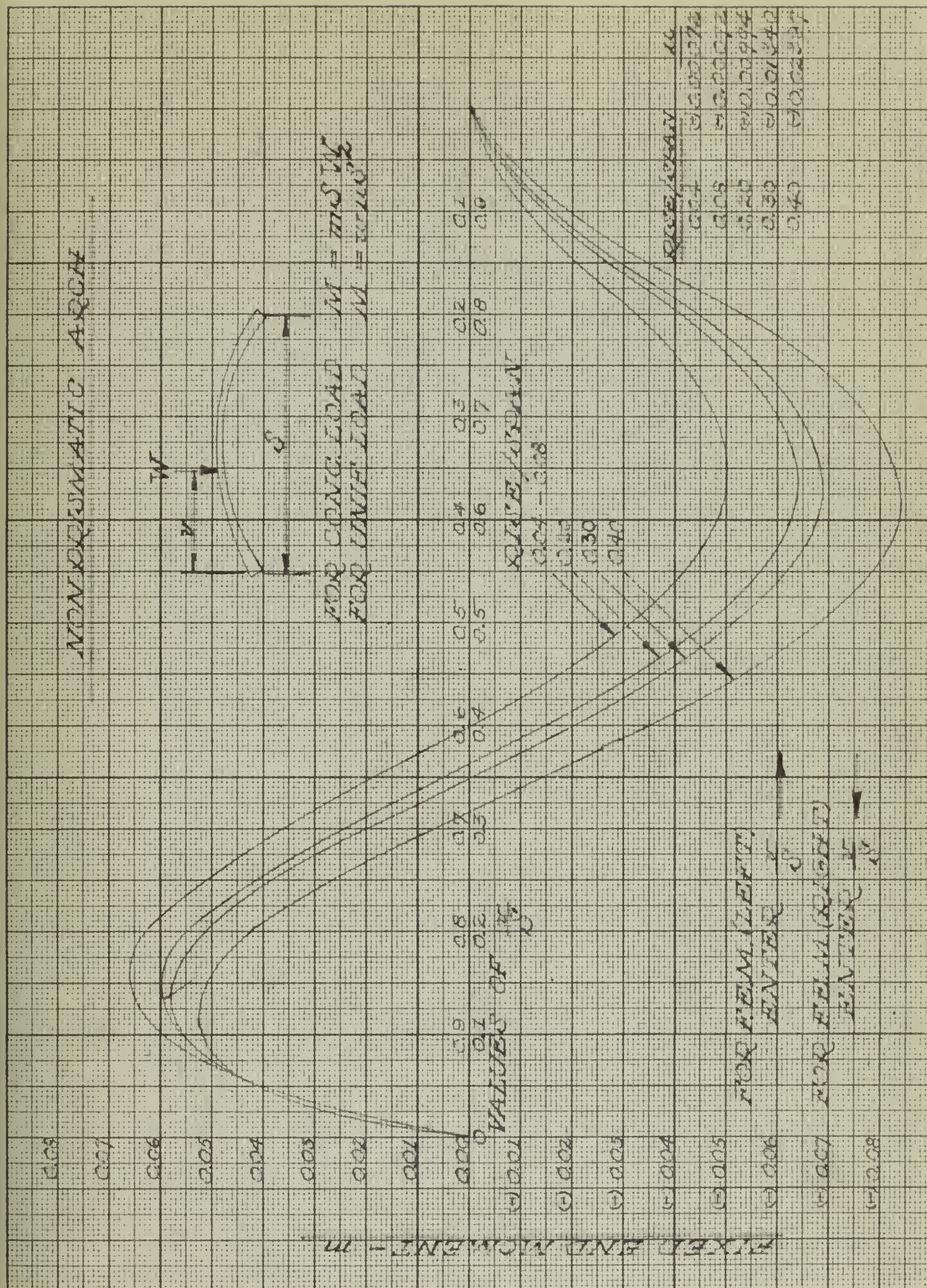
Run	Time (min)	Temperature (°C)	Pressure (mm Hg)
6	60.0	100.0	760.0
7	70.0	100.0	760.0
8	80.0	100.0	760.0
9	90.0	100.0	760.0
10	100.0	100.0	760.0

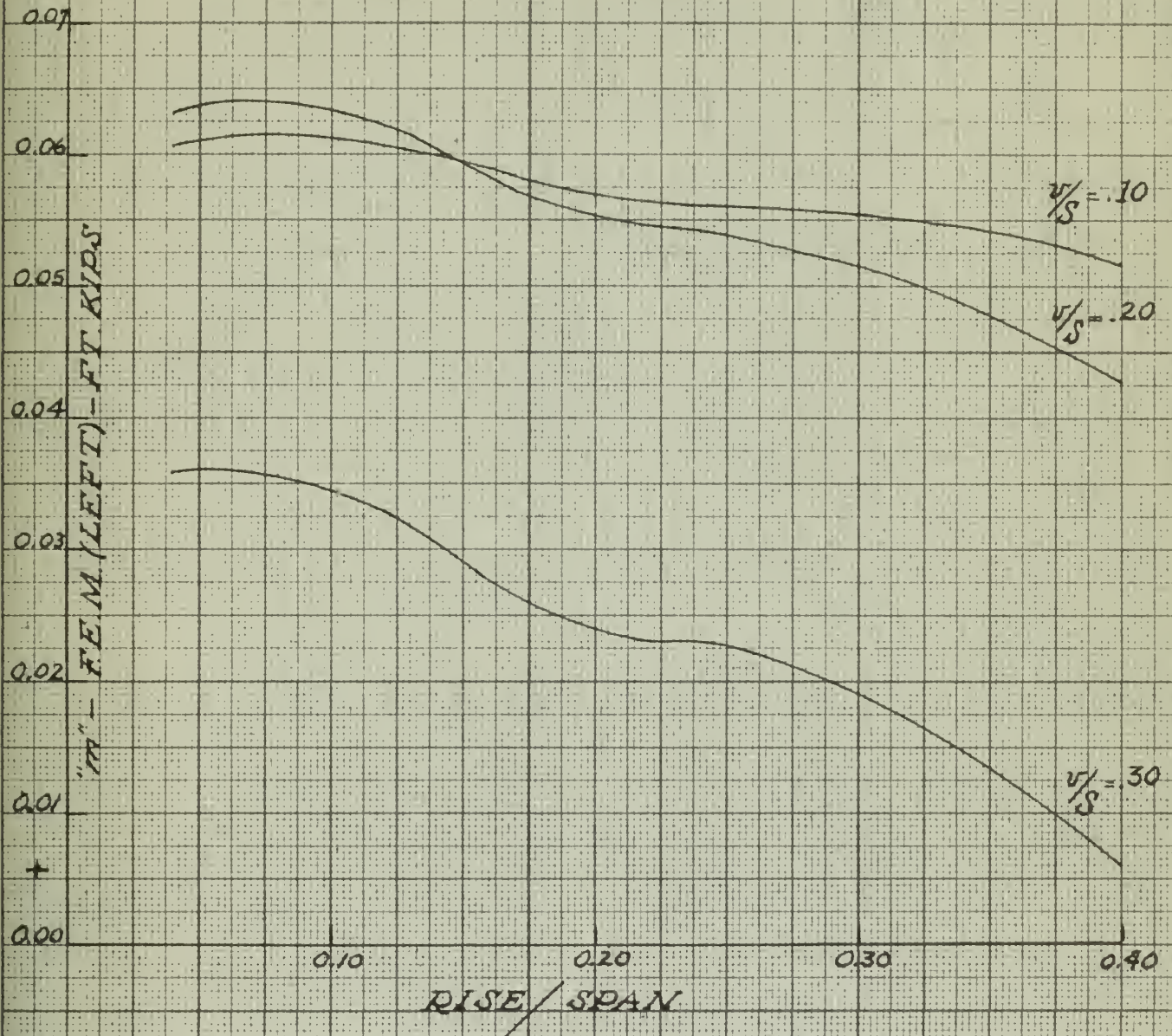
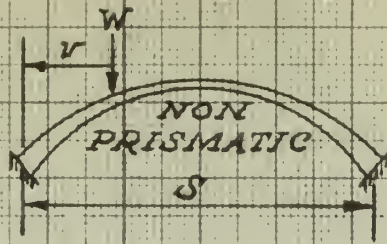
The values are given in the units indicated. The values are given in the units indicated.

Run	Time (min)	Temperature (°C)	Pressure (mm Hg)
11	110.0	100.0	760.0
12	120.0	100.0	760.0
13	130.0	100.0	760.0
14	140.0	100.0	760.0
15	150.0	100.0	760.0

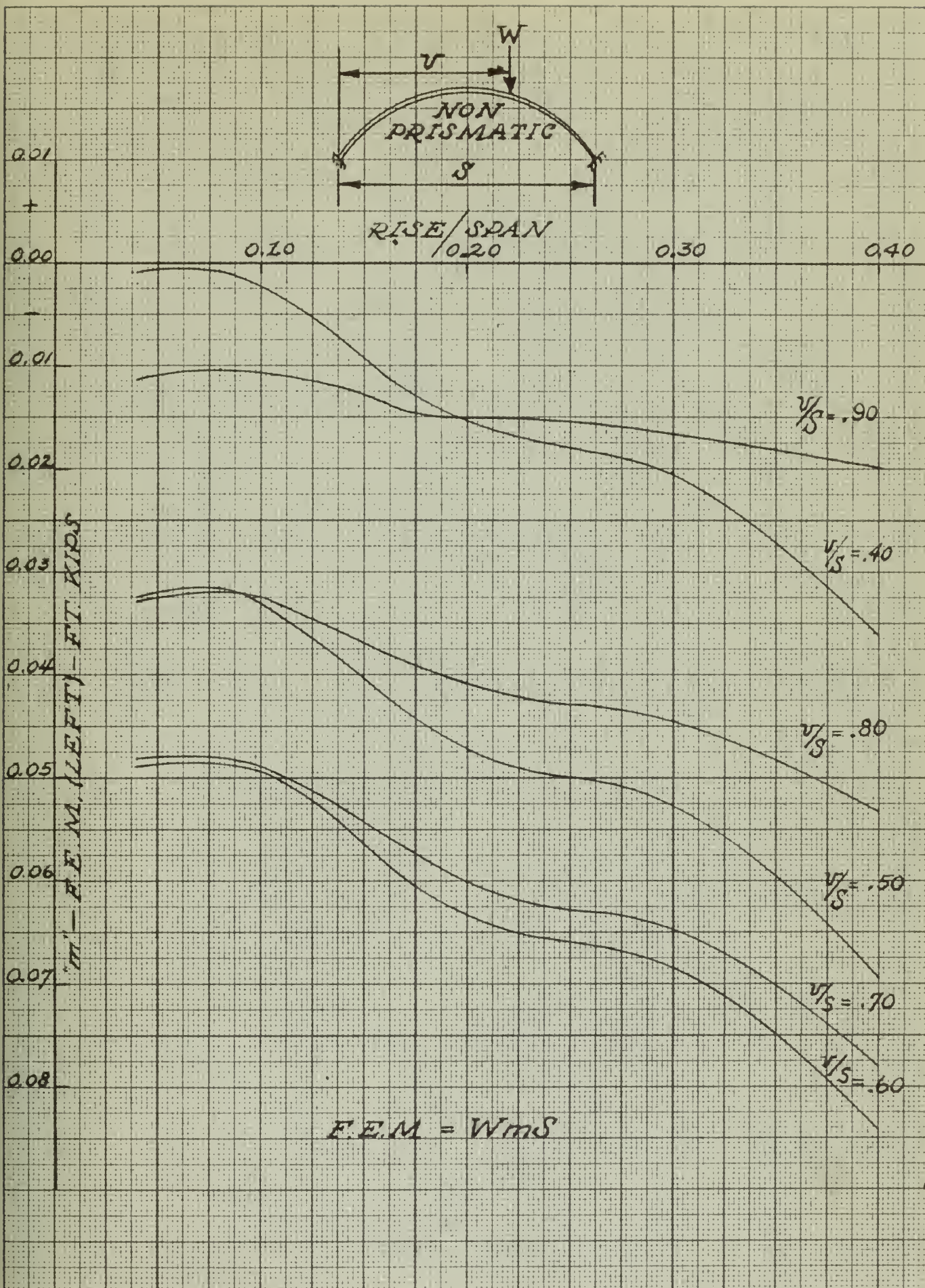
CURVES FOR NON-PRISMATIC ARCH

$$I_x = I_c \sec \theta$$





$$F.E.M. = WmS$$



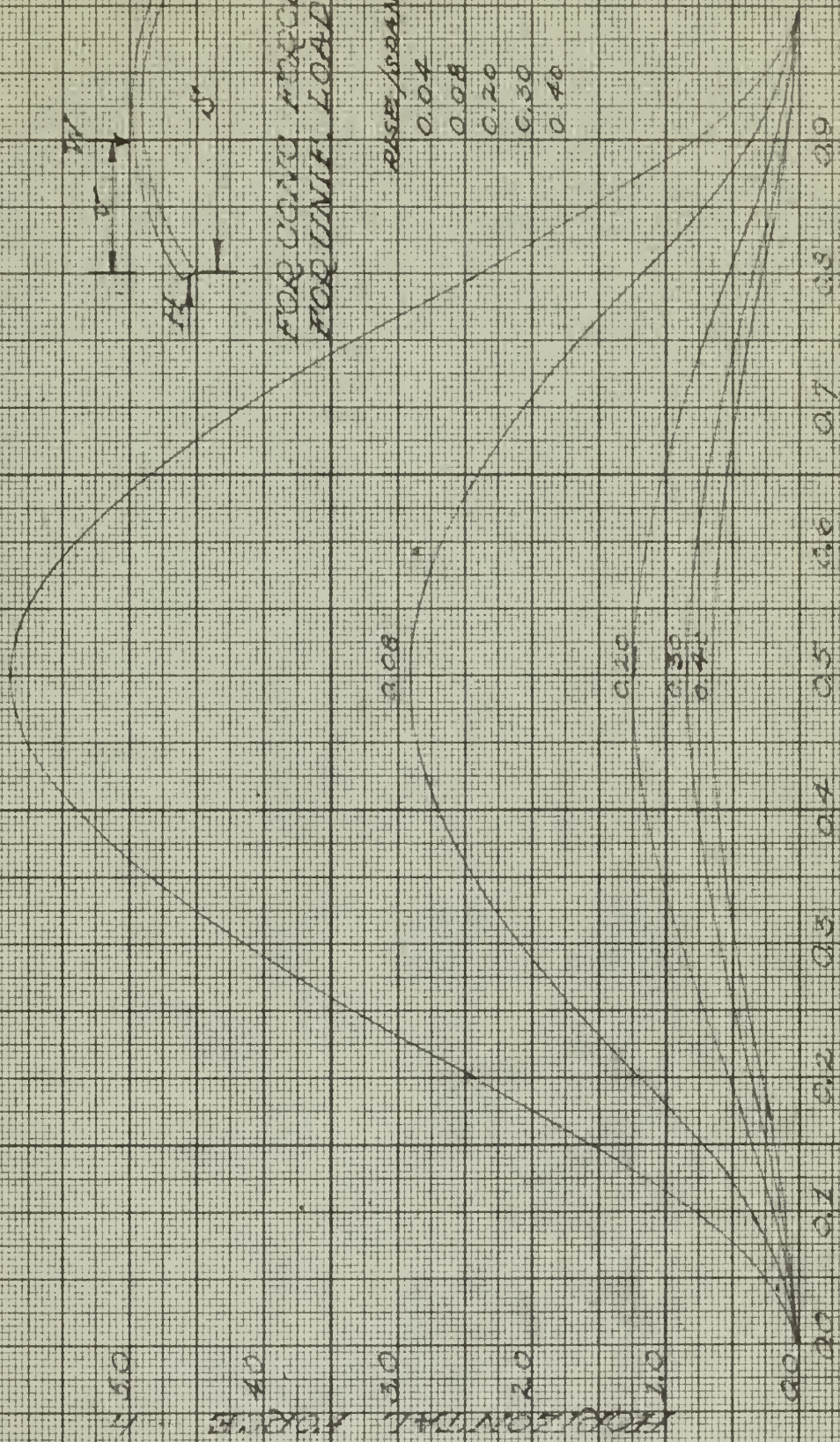
NON-DYNAMATIC ARCH

RISE/SPAN
0.04

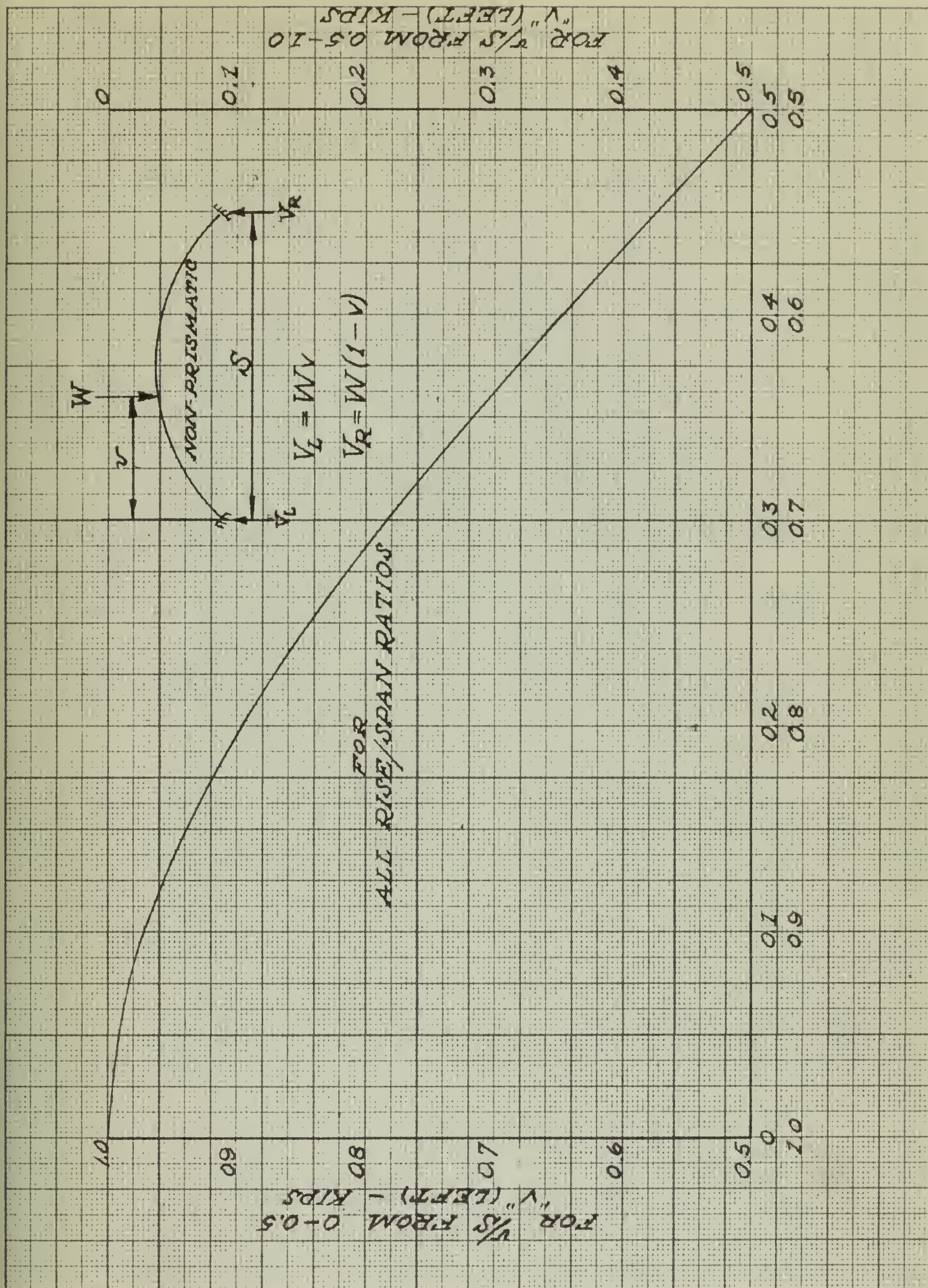


FOR CONG. FORCE $H=WH$
FOR UNIF. LOAD $H'=p \cdot x$

RISE/SPAN	P
0.04	3.1443
0.08	1.5608
0.20	0.6800
0.30	0.4562
0.40	0.3580



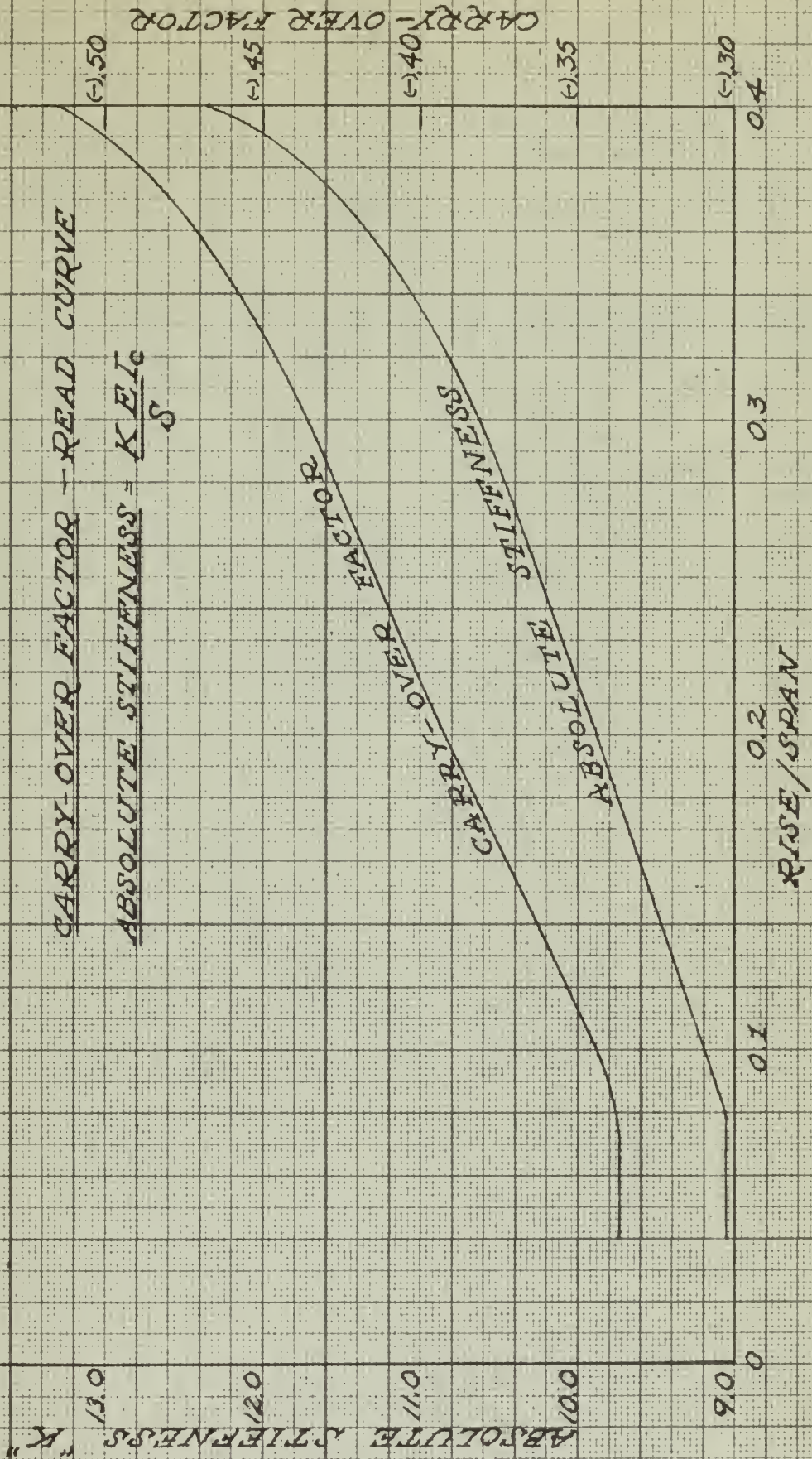
VALUES OF x/y

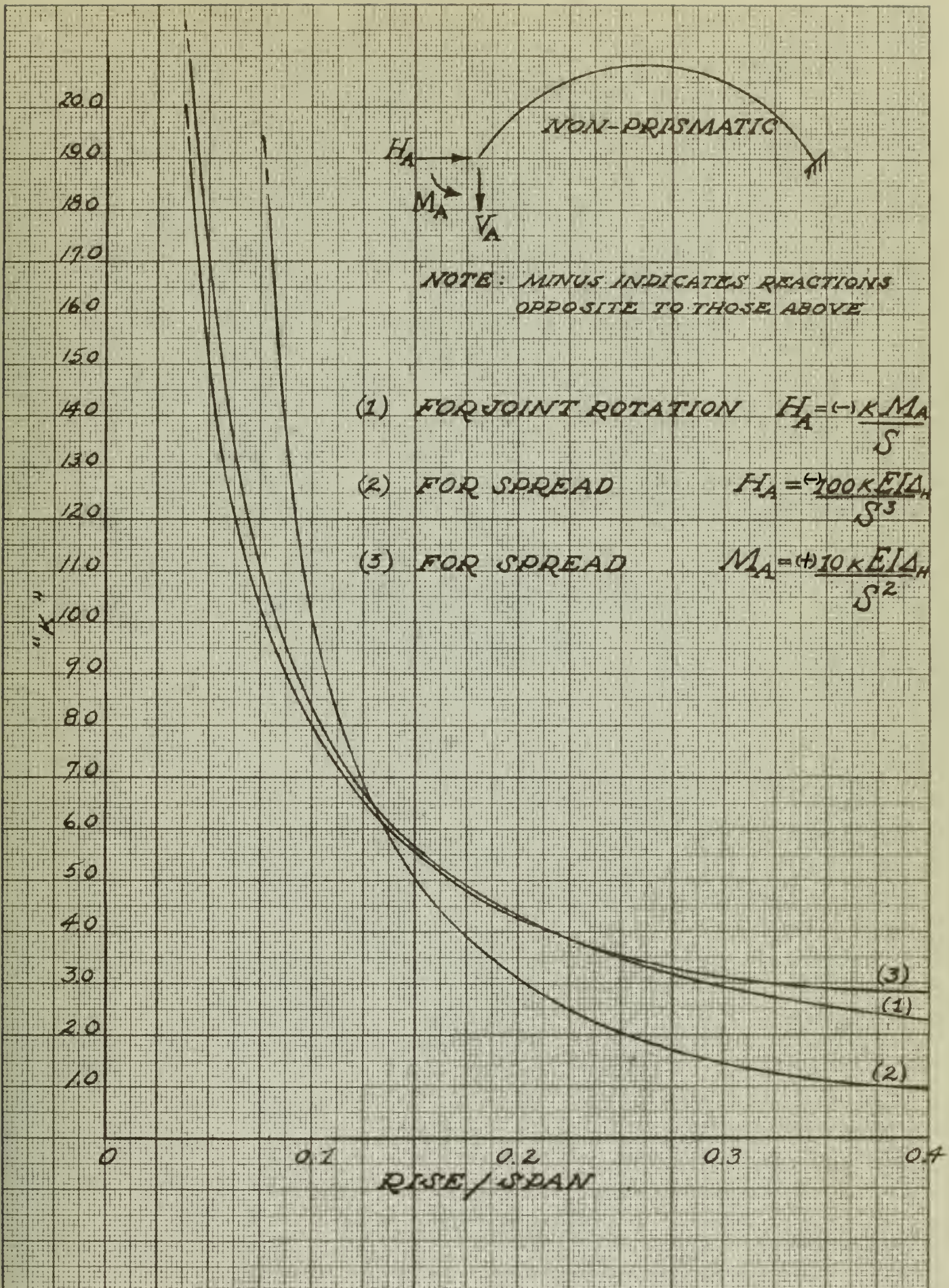


FOR NON-PRISMATIC ARCH

CARRY-OVER FACTOR - READ CURVE

$$\text{ABSOLUTE STIFFNESS} = \frac{KEI_c}{S}$$





RESULTS FOR NON-PRISMATIC ARCH

BASED ON A UNIT SPAN

TABULATION OF CONSTANTS FOR A NON-PRISMATIC MEMBER
 BASED ON A UNIT SPAN
 Rise/Span = 0.04

Load Point	Fea (Left) Ft.KIPS	Fea (Right) Ft.KIPS	V (Left)	H (Thrust)
1	+0.060585 S	-.011415 S	.972000	.764589
2	+0.063069 S	-.032931 S	.896000	2.431799
3	+0.035853 S	-.048147 S	.784000	4.162677
4	-.000922 S	-.048922 S	.648000	5.427620
5	-.032445 S	-.032445 S	.500000	5.896671

Rise/Span = 0.08

1	+0.051564 S	-.010436 S	.972000	.562680
2	+0.063975 S	-.032025 S	.896000	1.194386
3	+0.035576 S	-.048425 S	.784000	2.078643
4	-.000846 S	-.048846 S	.648000	2.702132
5	-.031606 S	-.031606 S	.500000	3.921515

Rise/Span = 0.20

1	+0.058325 S	-.015175 S	.972000	.175661
2	+0.055910 S	-.040790 S	.896000	.528900
3	+0.023923 S	-.060077 S	.784000	.494297
4	-.015405 S	-.063405 S	.648000	1.153284
5	-.047449 S	-.047449 S	.500000	1.253049

Rise/Span = 0.30

1	+0.055303 S	-.016697 S	.972000	.120171
2	+0.051587 S	-.044413 S	.896000	.357347
3	+0.019117 S	-.084864 S	.784000	.598050
4	-.020492 S	-.068492 S	.648000	.789252
5	-.052644 S	-.052644 S	.500000	.930758

Rise/Span = 0.40

1	+0.052034 S	-.019916 S	.972000	.096968
2	+0.042737 S	-.058263 S	.896000	.285925
3	+0.005919 S	-.078081 S	.784000	.473109
4	-.036356 S	-.084356 S	.648000	.604815
5	-.069383 S	-.069383 S	.500000	.651856

Tabulation of Results for Non-Prismatic Arch

<u>Rise/Span</u>	<u>Absolute Stiffness</u>	<u>Carry-over Factor</u>
0.04	9.051219 EI_c/s	(-) .3371054
0.08	9.062642 EI_c/s	(-) .3379414
0.20	9.879723 EI_c/s	(-) .3926988
0.30	10.829838 EI_c/s	(-) .4355514
0.40	12.350376 EI_c/s	(-) .5141846

With Joint A free to rotate, and a moment of M_a ft.kips applied at A, the following thrusts and shears are induced at A.

<u>Rise/Span</u>	<u>Thrust (KIPS)</u>	<u>Vertical Shear (KIPS)</u>
0.04	20.900806 $EI_c M_a/s$.6628946 $EI_c M_a/s$
0.08	10.421292 $EI_c M_a/s$.6320586 $EI_c M_a/s$
0.20	4.324337 $EI_c M_a/s$.6073012 $EI_c M_a/s$
0.30	2.916752 $EI_c M_a/s$.5644486 $EI_c M_a/s$
0.40	2.267353 $EI_c M_a/s$.4958152 $EI_c M_a/s$

When a spread of ΔH feet occurs, the following moments, shears and thrusts occur.

<u>Rise/Span</u>	<u>Thrust (KIPS)</u>	<u>Moment (Pt.KIPS)</u>	<u>Shear</u>
0.04	7065.0662 $EI_c \Delta H/s^3$	(-) 189.17764 $EI_c \Delta H/s^2$	0
0.08	1761.8736 $EI_c \Delta H/s^3$	(-) 94.44436 $EI_c \Delta H/s^2$	0
0.20	310.4368 $EI_c \Delta H/s^3$	(-) 42.72352 $EI_c \Delta H/s^2$	0
0.30	144.9938 $EI_c \Delta H/s^3$	(-) 31.00462 $EI_c \Delta H/s^2$	0
0.40	93.9058 $EI_c \Delta H/s^3$	(-) 28.00266 $EI_c \Delta H/s^2$	0

TABLE 1. SUMMARY OF DATA FOR THE 1970-71 FLOODING

STATION	DATE	WATER LEVEL (ft)	WIND (mph)	WAVE HEIGHT (ft)
1. STATION 1	10/1/70	10.0	10.0	10.0
2. STATION 2	10/2/70	10.0	10.0	10.0
3. STATION 3	10/3/70	10.0	10.0	10.0
4. STATION 4	10/4/70	10.0	10.0	10.0
5. STATION 5	10/5/70	10.0	10.0	10.0

NOTE: The data for the 1970-71 flooding are given in Table 1.

TABLE 2. SUMMARY OF DATA FOR THE 1971-72 FLOODING

TABLE 2

STATION	DATE	WATER LEVEL (ft)	WIND (mph)	WAVE HEIGHT (ft)
1. STATION 1	10/1/71	10.0	10.0	10.0
2. STATION 2	10/2/71	10.0	10.0	10.0
3. STATION 3	10/3/71	10.0	10.0	10.0
4. STATION 4	10/4/71	10.0	10.0	10.0
5. STATION 5	10/5/71	10.0	10.0	10.0

NOTE: The data for the 1971-72 flooding are given in Table 2.

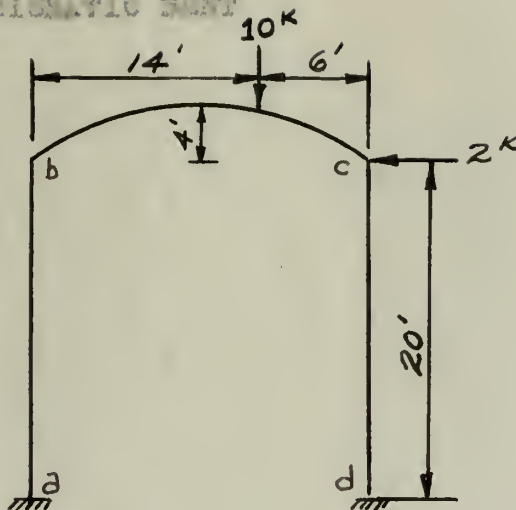
TABLE 3. SUMMARY OF DATA FOR THE 1972-73 FLOODING

STATION	DATE	WATER LEVEL (ft)	WIND (mph)	WAVE HEIGHT (ft)
1. STATION 1	10/1/72	10.0	10.0	10.0
2. STATION 2	10/2/72	10.0	10.0	10.0
3. STATION 3	10/3/72	10.0	10.0	10.0
4. STATION 4	10/4/72	10.0	10.0	10.0
5. STATION 5	10/5/72	10.0	10.0	10.0

APPLICATION OF RESULTS

PRISMATIC ARCH TEST

FOR A PRISMATIC JOINT



$$\text{Rise} = 4'$$

$$\frac{\text{Rise}}{\text{Span}} = \frac{4}{20} = .2$$

From Curves: $M_{bc}^f = (-).0465 \times 10^k \times 20 = (-) 9.30 \text{ ft.k.}$

$$M_{cb}^f = (+).0306 \times 10 \times 20 = (+) 6.12 \text{ ft.k.}$$

$$H_b = .320 \times 10 = 3.2^k \rightarrow$$

$$H_c = .320 \times 10 = 3.2^k \leftarrow$$

with a positive moment indicating tension on the top of member.

In moment distribution a moment that tends to rotate the joint clockwise is considered positive, therefore

$$M_{bc}^f = (-) 9.30 \text{ ft. k.} \quad M_{cb}^f = (-) 6.12 \text{ ft.k.}$$

The absolute stiffness from page 57 is $7.6950 \frac{EI}{L} = .38479 EI$

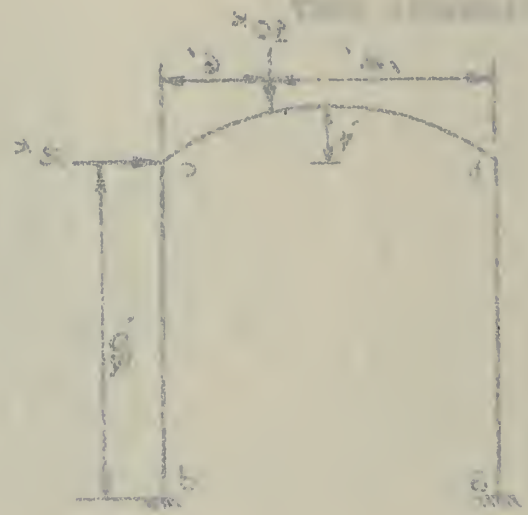
Use .3848 EI

The carry over factor is $(-).3489$

The absolute stiffness of the prismatic column members is

$$\frac{4EI}{L} = .2EI \text{ and the carry over factor is } (+).500.$$

FIG. 1. A SIMPLE ARCH



Let x be the horizontal distance from the left support to the point of interest, and let y be the vertical height of the arch at that point. Then the equation of the arch is given by:

$$y = \frac{4}{L^2} x(L-x)$$

where L is the total length of the arch, and x and y are the horizontal and vertical coordinates of any point on the arch.

The maximum height of the arch is reached at the center, where $x = L/2$. Substituting this value into the equation above, we find that the maximum height is $y = L/4$.

$$y = \frac{4}{L^2} x(L-x)$$

Let $x = L/2$, then $y = \frac{4}{L^2} \cdot \frac{L}{2} \cdot \frac{L}{2} = \frac{4}{L^2} \cdot \frac{L^2}{4} = 1$.

The area under the arch can be found by integrating the equation of the arch with respect to x . The area is given by:

$$A = \int_0^L y \, dx = \int_0^L \frac{4}{L^2} x(L-x) \, dx = \frac{4}{L^2} \left[Lx^2 - \frac{x^3}{3} \right]_0^L = \frac{4}{L^2} \left(L^3 - \frac{L^3}{3} \right) = \frac{4}{L^2} \cdot \frac{2L^3}{3} = \frac{8L}{3}$$

FOR PRISMATIC ARCH

Assuming rotation of the joints only, solve for the moments at all joints by moment distribution.

Joint	A		B		C		D
Member	AB	BA	BC		CB	CD	DC
K	.2	.2	.3848	BALANCING	.3848	.2	.2
$\frac{K}{\Sigma K}$	0	.342	.658	MOMENTS	.658	.342	0
COF	0	.5	-.3489	-.3489	-.3489	+.5	0
			-9.30		-6.12		
	+1.59	+3.18	+6.12	+6.12	-2.14		
			-1.90	+5.44	+5.44	+2.82	+1.41
	+0.33	+0.65	+1.25	+1.25	-0.44		
			-0.10	+0.29	+0.29	+0.15	+ .08
	+0.02	+0.03	+0.07	+0.07	-0.02		
				+0.01	+0.01	+0.01	
Net	+1.94	+3.86	-3.86		-2.98	+2.98	+1.49
Bal							
Mom				+7.44	+5.74		

A record of the balancing moments taken by the arch must be kept in order that the horizontal restraining force at the ends of the arched member may be determined. The magnitude of this restraining force depends on the properties of the arch under consideration.

The required horizontal restraining forces at the ends of the arch are made up of the following components:

1	2	3	4	5	6	7	8
100	100	100		100	100	100	100
90	90	90	0.90	90	90	90	90
80	80	80	0.80	80	80	80	80
70	70	70	0.70	70	70	70	70
60	60	60	0.60	60	60	60	60
50	50	50	0.50	50	50	50	50
40	40	40	0.40	40	40	40	40
30	30	30	0.30	30	30	30	30
20	20	20	0.20	20	20	20	20
10	10	10	0.10	10	10	10	10
0	0	0	0.00	0	0	0	0

(1) That force necessary to restrain the horizontal thrust at the ends of the arch caused by a load on the arch or caused by the tendency of the arch to spread.

(2) That force necessary to restrain the horizontal thrust at the ends of the arch as caused by the balancing moments.

(3) That force necessary to balance the shears in the column members.

(4) That force necessary to prevent sidesway when a horizontal load is applied to the bent.

Using the sign convention of moment distribution, a positive balancing moment at the left springing causes horizontal restraining forces at the two ends of the arch acting in directions away from each other. A positive balancing moment at the right springing causes equal horizontal restraining forces acting in directions toward each other.

Enclosed for the Secretary of the Board of Education
are two copies of the report of the Committee on
the subject of the proposed new school law.
Very respectfully,
J. B. [Signature]

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FOR PRISMATIC BENT

The artificial joint restraints (AJR) required to prevent the original bent from deflecting due to sidesway or spread are equal to the horizontal restraining forces at the ends of the arch. ($H = 4.2168 \frac{W}{3} = .21084$ See Page 57)

HRF Caused by:

	$\frac{HRF_1}{20}$ \rightarrow	$\frac{HRF_2}{20}$
(a) Load on Arch	(+)8.20	(-)8.20
(b) Balancing Moment at B (-0.21084)(7.44)	(-)1.57	(+)1.57
(c) Balancing Moment at C (0.21084)(5.74)	(+)1.21	(-)1.21
(d) Shear in Member AB $\frac{(1.94 + 3.86)}{20}$	(+)0.22	
(e) Shear in Member CD $\frac{(1.49 + 2.98)}{20}$		(+)0.22
(f) 2 Kip Load at C		(+)2.00
	AJR_1 (+)8.13	AJR_2 (-)5.62

Let x = AJR resisting spread
Let y = AJR resisting sidesway
then

$$\begin{array}{lcl}
 \begin{array}{c} \xrightarrow{x} \text{---} \text{---} \text{---} \xleftarrow{x} \\ \xrightarrow{y} \text{---} \text{---} \text{---} \xrightarrow{y} \end{array} & \begin{array}{l} x + y = 8.13 \\ x - y = 5.62 \\ 2x = 13.75 \\ x = 6.875^k \\ y = 1.255^k \end{array}
 \end{array}$$

Thus the AJR_{sp} required to resist spread = $6.875^k \rightarrow \leftarrow$
and the AJR_{ss} required to resist sidesway = $1.255^k \rightarrow$

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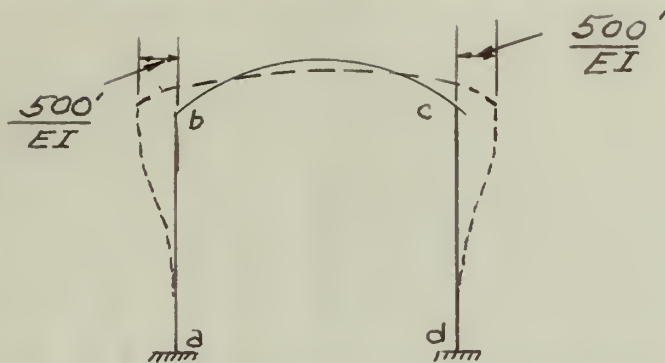
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1
10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1
10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1
10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1
10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1
10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1
10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1
10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1
10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1
10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1	10.11.1
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Assume a spread of 1000'/EI feet, allowing no joint rotation.



The following reactions at the ends of the arch result from the above spread of $\frac{1000'}{EI}$

From Page 57

$$H_b = 245.8247 \frac{EI \Delta H}{s^3} = \frac{245.8247 EI}{(20)^3} \times \frac{1000}{EI} = 30.7281 \leftarrow$$

$$H_c = 30.7281 \rightarrow$$

$$M_{bc}^f = (+)32.45204 \frac{EI \Delta H}{s^2} = \frac{32.45204 EI}{(20)^2} \times \frac{1000}{EI} = (+)81.1301$$

$$M_{cb}^f = (+)81.130$$

The above moments indicate tension on the top side, hence for moment distribution sign convention:

$$M_{bc}^f = (+)81.1301 \text{ ft.k.} \quad M_{cb}^f = (-)81.1301 \text{ ft.k.}$$

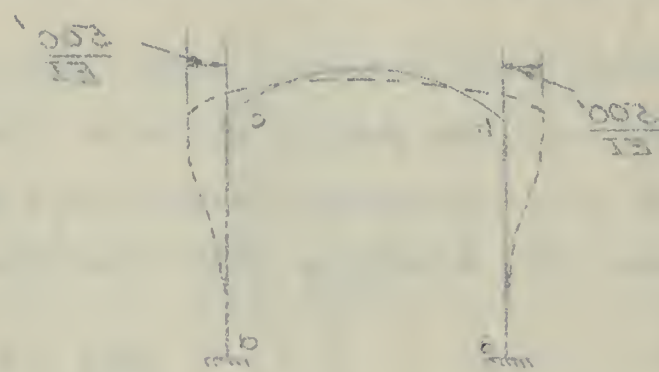
The moments induced in the ends of the column members are equal to $\frac{6EI\Delta H}{L^2}$

$$= \frac{6EI}{(20)^2} (500) = 7.5 \text{ ft.kips}$$

$$\text{and } M_{ab}^f = (-)7.5 = M_{ba}^f$$

$$M_{cd}^f = (+)7.5 = M_{dc}^f$$

Figure 1 shows the typical cross-section of the bridge deck.



The following table gives the data for the bridge deck.

Table 1. Bridge Deck Data

27

$$2000.00 = \frac{1000}{10} = \frac{1000 \times 100}{1000} = \frac{100000}{1000} = 100.00$$

$$1000.00 = \frac{1000}{10} = \frac{1000 \times 100}{1000} = \frac{100000}{1000} = 100.00$$

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The above table gives the data for the bridge deck.

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$$1000.00 = \frac{1000}{10} = \frac{1000 \times 100}{1000} = \frac{100000}{1000} = 100.00$$

Solving for moments at the joints caused by the assumed spread

Joint	A		B			C		D
Member	AB	BA	BC		BALANCING	CB	CD	DC
K	.2	0.2	0.3848		MOMENTS	0.3848	0.2	.2
$\sum K$	0	0.348	0.658			0.658	0.348	0
CJP	0	+ 0.5	-0.3489		-0.3489	-0.3489	+0.5	0
Fix	- 7.50	- 7.50	+81.13			-81.13	+ 7.50	+ 7.50
	-12.59	-25.18	-48.45	-48.45		+16.90		
			-13.02		+37.33	+37.33	+19.40	+ 9.70
	+ 2.23	+ 4.45	+ 8.57	+ 8.57		- 2.99		
			- 0.62		+ 1.97	+ 1.97	+ 1.02	+ 0.81
	+ 0.12	+ 0.24	+ 0.45	+ 0.45		- 0.16		
					+ 0.11	+ 0.11	+ 0.05	+ 0.03
Map	-17.74	-27.99	+27.99			-27.97	+27.97	+17.74
bal.								
Mom.					-39.43	+39.41		

The consistent joint force (CJP) required at B and C to maintain the assumed spread must equal the horizontal restraining forces at B and C.

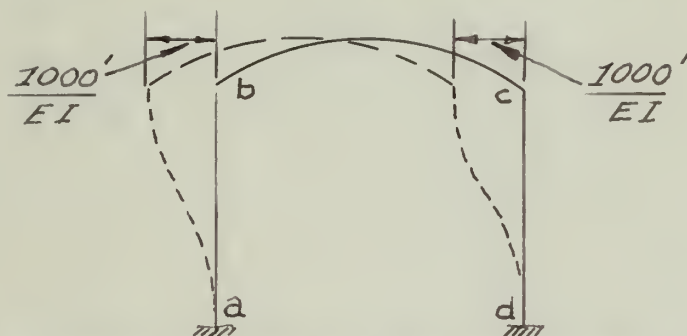
HMF Caused by:		HRFL	HRFR
(1) spread in the arch		-39.73	+39.73
(2) Balancing moment at joint B (0.21084) (39.43)		+ 8.31	- 8.31
(3) Balancing moment at joint C (0.21084) (39.41)		+ 8.31	- 8.31
(4) Shear in member AB	$\frac{17.74 + 27.99}{20} = 45.73$	- 2.29	
(5) Shear in member BC	$\frac{17.74 + 27.99}{20} =$		+ 2.29
	CJP _{sp}	-16.40 ^k	16.40 ^k

Since a consistent joint force of 16.40^k at each end of the arch acting away from each other will produce the assumed spread of $\frac{1000}{EI}$ ft., and since an artificial joint restraint of 6.875 kips was required to restrain the original structure from spreading under the action of the 10 kip vertical load, the actual spread will cause moments at the various joints equal to

$$\frac{AJR_{sp}}{CJR_{sp}} (M_{sp} \text{ for } \frac{1000}{EI}) = \frac{6.875}{16.40} M_{sp} = .4192 M_{sp}$$

SIDESWAY

Assume a sidesway of $\frac{1000}{EI}$ feet to the left.



The moments induced in the column members as a result of the assumed sidesway equals

$$\frac{GEI}{L^2} = \frac{3EI}{(20)^2 EI} = 15 \text{ fk.}$$

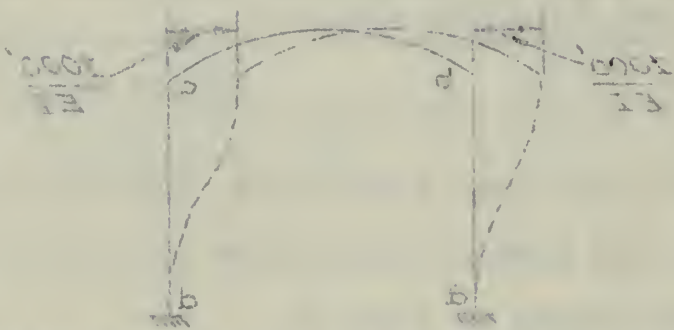
$$M_{ab}^f = (-)15 \text{ fk.}, M_{ba}^f = -15 \text{ fk.}, M_{cd}^f = -15 \text{ fk.}, M_{dc}^f = -15 \text{ fk.}$$

These two conditions being known, the value of θ can be found by the following method. Let θ be the angle between the vertical and the line of action of the force. Then the horizontal component of the force is $P \sin \theta$ and the vertical component is $P \cos \theta$. The horizontal component must be equal to the weight of the body, and the vertical component must be equal to the reaction at the pivot. These two conditions can be written as follows:

$$P \sin \theta = W \quad \text{and} \quad P \cos \theta = R$$

Dividing the first equation by the second, we get

$$\tan \theta = \frac{W}{R}$$



The reaction force R is the force exerted by the pivot on the body. It is equal in magnitude to the weight W of the body, and acts in the opposite direction.

$$R = W$$

$$\tan \theta = \frac{W}{W} = 1 \quad \therefore \theta = 45^\circ$$

Solving for the moments at the joints caused by the assumed sidesway.

Joint	A	B				C	D	
Member	AB	BA	BC			CB	CD	DC
K	0.2	0.2	0.3848			0.3848	0.2	0.2
$\frac{IK}{\Sigma K}$	0	0.342	0.658	BALANCING MOMENTS		0.658	0.342	0
CJF	0	+ 0.5	-0.3489	-0.3489		-0.3489	+ 0.5	0
PM	-15.00	-15.00					-15.00	-15.00
	+ 2.57	+ 5.13	+ 9.87	+ 9.87		- 3.44		
			- 4.23		+12.15	+12.13	+ 6.31	+ 3.16
	+ 0.73	+ 1.45	+ 2.78	+ 2.78		- 0.27		
			- 0.22		+ 0.64	+ 0.64	+ 0.33	+ 0.17
	+ 0.04	+ 0.08	+ 0.14	+ 0.14				
M _{ss}	-11.66	- 8.34	+ 8.34			+ 8.36	- 8.36	-11.67
Bal.								
Moms.				+12.79	+12.77			

The consistent joint force (CJF) required at B and C to maintain the assumed sidesway must equal the horizontal restraining forces at B and C.

HRF caused by:

- (1) Balancing Moment at Joint B
(0.21034)(12.79)
- (2) Balancing Moment at Joint C
(0.21084)(12.77)
- (3) Shear in Member AB $\frac{11.66 + 8.34}{20}$
- (4) Shear in Member CD $\frac{11.67 + 8.36}{20}$

CJF_{ss}

HRF _L	$\xrightarrow{+}$	HRF _R
(-)2.69		(+)2.69
(+)2.69		(-)2.69
(-)1.00		
		(-)1.00
(-)1.00 ^k	$\xleftarrow{-}$	(-)1.00 ^k
		$\xleftarrow{-}$

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TABLE I				TABLE II			
RESULTS OF EXPERIMENT I				RESULTS OF EXPERIMENT II			
Time	Temp.	Pressure	Volume	Time	Temp.	Pressure	Volume
0.0	20.0	100.0	100.0	0.0	20.0	100.0	100.0
1.0	20.5	101.0	101.0	1.0	20.5	101.0	101.0
2.0	21.0	102.0	102.0	2.0	21.0	102.0	102.0
3.0	21.5	103.0	103.0	3.0	21.5	103.0	103.0
4.0	22.0	104.0	104.0	4.0	22.0	104.0	104.0
5.0	22.5	105.0	105.0	5.0	22.5	105.0	105.0
6.0	23.0	106.0	106.0	6.0	23.0	106.0	106.0
7.0	23.5	107.0	107.0	7.0	23.5	107.0	107.0
8.0	24.0	108.0	108.0	8.0	24.0	108.0	108.0
9.0	24.5	109.0	109.0	9.0	24.5	109.0	109.0
10.0	25.0	110.0	110.0	10.0	25.0	110.0	110.0
11.0	25.5	111.0	111.0	11.0	25.5	111.0	111.0
12.0	26.0	112.0	112.0	12.0	26.0	112.0	112.0
13.0	26.5	113.0	113.0	13.0	26.5	113.0	113.0
14.0	27.0	114.0	114.0	14.0	27.0	114.0	114.0
15.0	27.5	115.0	115.0	15.0	27.5	115.0	115.0
16.0	28.0	116.0	116.0	16.0	28.0	116.0	116.0
17.0	28.5	117.0	117.0	17.0	28.5	117.0	117.0
18.0	29.0	118.0	118.0	18.0	29.0	118.0	118.0
19.0	29.5	119.0	119.0	19.0	29.5	119.0	119.0
20.0	30.0	120.0	120.0	20.0	30.0	120.0	120.0

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TABLE III				TABLE IV			
RESULTS OF EXPERIMENT III				RESULTS OF EXPERIMENT IV			
Time	Temp.	Pressure	Volume	Time	Temp.	Pressure	Volume
0.0	20.0	100.0	100.0	0.0	20.0	100.0	100.0
1.0	20.5	101.0	101.0	1.0	20.5	101.0	101.0
2.0	21.0	102.0	102.0	2.0	21.0	102.0	102.0
3.0	21.5	103.0	103.0	3.0	21.5	103.0	103.0
4.0	22.0	104.0	104.0	4.0	22.0	104.0	104.0
5.0	22.5	105.0	105.0	5.0	22.5	105.0	105.0
6.0	23.0	106.0	106.0	6.0	23.0	106.0	106.0
7.0	23.5	107.0	107.0	7.0	23.5	107.0	107.0
8.0	24.0	108.0	108.0	8.0	24.0	108.0	108.0
9.0	24.5	109.0	109.0	9.0	24.5	109.0	109.0
10.0	25.0	110.0	110.0	10.0	25.0	110.0	110.0
11.0	25.5	111.0	111.0	11.0	25.5	111.0	111.0
12.0	26.0	112.0	112.0	12.0	26.0	112.0	112.0
13.0	26.5	113.0	113.0	13.0	26.5	113.0	113.0
14.0	27.0	114.0	114.0	14.0	27.0	114.0	114.0
15.0	27.5	115.0	115.0	15.0	27.5	115.0	115.0
16.0	28.0	116.0	116.0	16.0	28.0	116.0	116.0
17.0	28.5	117.0	117.0	17.0	28.5	117.0	117.0
18.0	29.0	118.0	118.0	18.0	29.0	118.0	118.0
19.0	29.5	119.0	119.0	19.0	29.5	119.0	119.0
20.0	30.0	120.0	120.0	20.0	30.0	120.0	120.0

Since a consistent joint force of 1.00 kips at each end of the arch acting toward the left will produce the assumed sideway of $\frac{1000}{11}$ feet, and since an artificial joint restraint of 1.255 kips was required to prevent sideway in the original structure under the action of the 10 k vertical load, the actual sideway will cause movements at the various joints equal to

$$\frac{AJR_{ss}}{CJP_{ss}} (M_{ss} \text{ for } \frac{1000}{11}) = \frac{1.255}{1.00} M_{ss} = 1.255 M_{ss}$$

TABULATION OF RESULTS

Factor	M _{ab}	M _{ba}	M _{bc}	M _{cb}	M _{cd}	M _{dc}
M _P	(+) 1.94	(+) 3.86	(-) 3.86	(-) 2.98	(+) 2.98	(+) 1.49
.4192 M _{sp}	(-) 7.44	(-) 11.73	(+) 11.73	(-) 11.73	(+) 11.73	(+) 7.44
1.255 M _{ss}	(-) 14.63	(-) 10.47	(+) 10.47	(+) 10.49	(-) 10.49	(-) 14.65
Final Internal Moments	(-) 20.13	(-) 18.34	(+) 18.34	(-) 4.22	(+) 4.22	(-) 5.72

$$H_a = \frac{20.13 + 18.34}{20} = 1.925 \text{ k} \rightarrow$$

$$H_d = \frac{(+4.22 - 5.72)}{20} = \frac{(-1.50)}{20} = .075 \rightarrow$$

Cutting the Bent at A

$$V_a(20) + 20.13 - 60 - 40 = (-) 5.72$$

$$V_a = \frac{(-) 20.13 + 60 + 40 - 5.72}{20} = \frac{74.15}{20} = 3.7075 \uparrow$$

Cutting the Bent at B

$$(-) V_d(20) + 5.72 - 40 + 140 = (-) 20.13$$

$$V_d = \frac{(-) 20.13 - 140 + 40 - 5.72}{(-) 20} = \frac{(-) 125.85}{(-) 20} = 6.2925 \uparrow$$

The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. It is shown that $f(x)$ is a continuous function and that it satisfies the differential equation $f'(x) = f(x)$. The second part of the paper is devoted to the study of the properties of the function $g(x)$ defined by the equation $g(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cos \frac{\pi n}{2}$. It is shown that $g(x)$ is a continuous function and that it satisfies the differential equation $g'(x) = -g(x)$.

$$\frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$$

x	$f(x)$	$g(x)$
0	1	1
1	e	$\frac{e}{2}$
-1	e^{-1}	$\frac{e^{-1}}{2}$
i	e^i	$\frac{e^i}{2}$
$-i$	e^{-i}	$\frac{e^{-i}}{2}$

$$\frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \cos \frac{\pi n}{2} \right) = \sum_{n=0}^{\infty} \frac{x^{n-1}}{(n-1)!} \cos \frac{\pi n}{2} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \cos \frac{\pi n}{2} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cos \frac{\pi(n+1)}{2} = -g(x)$$

$$\frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \sin \frac{\pi n}{2} \right) = \sum_{n=0}^{\infty} \frac{x^{n-1}}{(n-1)!} \sin \frac{\pi n}{2} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \sin \frac{\pi n}{2} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sin \frac{\pi(n+1)}{2} = h(x)$$

$$\frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$$

$$\frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \cos \frac{\pi n}{2} \right) = \sum_{n=0}^{\infty} \frac{x^{n-1}}{(n-1)!} \cos \frac{\pi n}{2} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \cos \frac{\pi n}{2} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cos \frac{\pi(n+1)}{2} = -g(x)$$

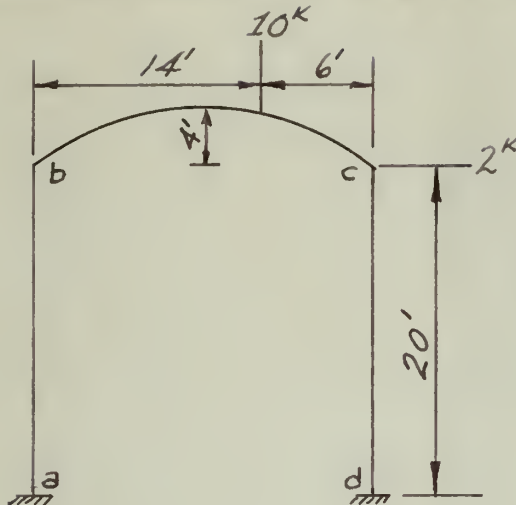
$$\frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \sin \frac{\pi n}{2} \right) = \sum_{n=0}^{\infty} \frac{x^{n-1}}{(n-1)!} \sin \frac{\pi n}{2} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \sin \frac{\pi n}{2} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sin \frac{\pi(n+1)}{2} = h(x)$$

$$\frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$$

APPLICATION OF RESULTS

NON-PRISMATIC AREA LIGHT

FOR A NON-PRISMATIC BENT ($I_x = I_c \sec \theta$)



The application of the method of moment distribution to the non-prismatic is the same as for the prismatic. This problem is worked primarily to show the relative values of moments and reactions developed under the same loading on a bent whose rise to span ratio and overall dimensions are the same as for the prismatic arch bent but whose I_{crown} is equal to the $I_{\text{prismatic}}$ multiplied by $\cos \theta_a = I \times .72415$

From Curves:

$$M_{bc}^f = (-).0601 \times 10^k \times 20' = (-)12.02 \text{ ft.k.}$$

$$M_{cb}^f = (+).0239 \times 10 \times 20 = (+)4.78 \text{ ft.k.}$$

$$H_b = .8943 \times 10 = 8.943 \text{ k} \rightarrow$$

$$H_c = .8943 \times 10 = 8.943 \text{ k} \leftarrow$$

For Moment Distribution Sign Convention

$$M_{bc}^f = (-)12.02 \text{ ft.k.} \quad M_{cb}^f = (-)4.78 \text{ ft.k.}$$

Absolute Stiffness of:

$$\text{Arch (Page 66)} \quad \frac{9.87973 EI_c}{S} = \frac{9.87973 (.72415) EI}{20} = .35772 EI$$

$$\text{Column Members} \quad \frac{4EI}{L} = 0.2 EI$$

Carry-over factor of:

Arch (Page 66) = $(-).3927$ Column Members = $(+).50$

Assuming rotation of joints only,

Joint	A	B			C	D	
Member	AB	BA	BC		CB	CD	CC
K	0.2	0.2	.358	BALANCING	0.358	0.2	0.2
$\frac{K}{\sum K}$	0	0.358	0.642	MOMENTS	0.642	0.358	0
COF	0	$+ 0.5$	-0.393	-0.393	-0.393	$+ 0.5$	0
FIX			-12.02		$- 4.78$		
	$+ 2.15$	$+ 4.30$	$+ 7.72$	$+ 7.72$	$- 5.03$		
			$- 1.97$		$+ 5.01$	$+ 3.90$	$+ 1.40$
	$+ 0.36$	$+ 0.71$	$+ 1.26$	$+ 1.26$	$- 0.50$		
			$- 0.15$		$+ 0.32$	$+ 0.15$	$+ 0.09$
	$+ 0.03$	$+ 0.05$	$+ 0.08$	$+ 0.08$	$- 0.03$		
					$+ 0.02$	$+ 0.02$	$+ 0.01$
M_r	$+ 2.54$	$+ 5.06$	$- 5.03$		$- 2.99$	$+ 2.99$	$+ 1.49$
Sal.							
Mom.			$+ 2.08$	$+ 5.36$			

The artificial joint restraints= RJP's:

RJP Caused by:

	$\frac{RJP_L}{20}$	$\frac{RJP_R}{20}$
(1) Load on Arch	$(+)5.94$	$(-)5.94$
(2) Balancing Moment at B $(.21622)(9.06)$	$(-)1.96$	$(+)1.96$
(3) Balancing Moment at C $(.21622)(5.35)$	$(+)1.16$	$(-)1.16$
(4) Shear in AB $\frac{2.54 + 5.06}{20}$	$(+)0.38$	
(5) Shear in CD $\frac{2.99 + 1.49}{20}$		$(+)0.22$
(6) 2 kip load at C		$(+)2.00$
	$(+)8.52$	$(-)5.92$

Let x = AJR resisting spread

y = AJR resisting sideways

$$x + y = 8.52$$

$$x - y = 5.92$$

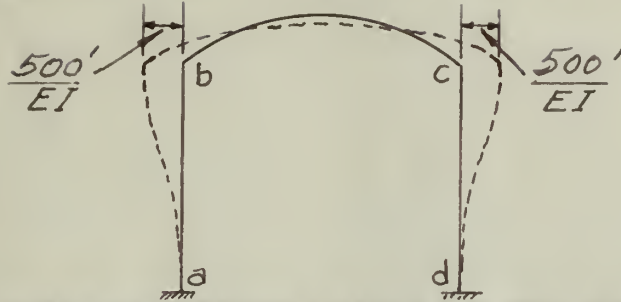
$$2x = 14.44$$

$$x = 7.22k = \text{AJR resisting spread (sp)}$$

$$y = 1.30k = \text{AJR resisting sideways (sa)}$$



Assuming a spread of 1000/EI feet, allowing no joint rotation.



The following reactions result at the area ends.

From Page 66

$$H_b = 310.4368 \frac{EI_c \Delta H}{S^3} = \frac{310.4368}{(20)^3} EI \times (.72415) \times \frac{1000}{EI} = 23.1006 \leftarrow$$

$$H_c = 23.1006 \rightarrow$$

$$M_{bc}^f = 42.7235 \frac{EI_c \Delta H}{S^2} = \frac{42.7235}{(20)^2} EI (.72415) \times \frac{1000}{EI} = (+)77.345$$

$$M_{cb}^f = (+)77.345$$

For Moment Distribution Sign Convention

$$M_{bc}^f = (+)77.345$$

$$M_{cb}^f = (-)77.345$$

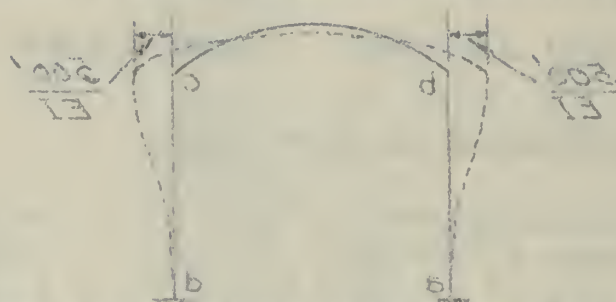
The moments induced at the ends of the column members are equal to $\frac{6EI\Delta H}{L^2}$

$$= \frac{6EI (500)}{(20)^2 EI} = 7.5 \text{ ft.kips}$$

$$\text{and } M_{ab}^f = (-)7.5 \text{ ft.kips} = M_{ba}^f$$

$$M_{cd}^f = (+)7.5 \text{ ft. kips} = M_{dc}^f$$

1. A beam of length \$L\$ is supported at both ends by vertical supports. A uniformly distributed load of intensity \$q\$ is applied over the entire length of the beam. The beam has a constant flexural rigidity \$EI\$.



The deflection curve is given by the equation:

From page 10

$$\delta_c = \frac{qL^4}{24EI} \left(\frac{x}{L} \right)^2 \left(1 - \frac{x}{L} \right)^2 = \frac{qL^4}{24EI} \left(\frac{x}{L} \right)^2 \left(1 - \frac{x}{L} \right)^2$$

$$\delta_d = \frac{qL^4}{24EI} \left(\frac{x}{L} \right)^2 \left(1 - \frac{x}{L} \right)^2$$

$$\delta_c = \frac{qL^4}{24EI} \left(\frac{x}{L} \right)^2 \left(1 - \frac{x}{L} \right)^2 = \frac{qL^4}{24EI} \left(\frac{x}{L} \right)^2 \left(1 - \frac{x}{L} \right)^2$$

$$\delta_c = \frac{qL^4}{24EI} \left(\frac{x}{L} \right)^2 \left(1 - \frac{x}{L} \right)^2$$

The deflection curve is given by the equation:

$$\delta_c = \frac{qL^4}{24EI} \left(\frac{x}{L} \right)^2 \left(1 - \frac{x}{L} \right)^2 \quad \delta_d = \frac{qL^4}{24EI} \left(\frac{x}{L} \right)^2 \left(1 - \frac{x}{L} \right)^2$$

The maximum deflection is at the center of the beam:

$$\delta_{max} = \frac{qL^4}{24EI}$$

$$\delta_c = \frac{qL^4}{24EI} \left(\frac{x}{L} \right)^2 \left(1 - \frac{x}{L} \right)^2 = \frac{qL^4}{24EI} \left(\frac{x}{L} \right)^2 \left(1 - \frac{x}{L} \right)^2$$

$$\delta_d = \frac{qL^4}{24EI} \left(\frac{x}{L} \right)^2 \left(1 - \frac{x}{L} \right)^2 = \frac{qL^4}{24EI} \left(\frac{x}{L} \right)^2 \left(1 - \frac{x}{L} \right)^2$$

$$\delta_c = \frac{qL^4}{24EI} \left(\frac{x}{L} \right)^2 \left(1 - \frac{x}{L} \right)^2 = \frac{qL^4}{24EI} \left(\frac{x}{L} \right)^2 \left(1 - \frac{x}{L} \right)^2$$

Solving for moments at the joints caused by the assumed spread

Joint	A		B			C		D
Member	AB	BA	BC	BALANCING		CB	CD	DC
K	0.2	0.2	0.358	BALANCING		0.358	0.2	0.2
$\frac{K}{\sum K}$	0	0.358	0.642			0.642	0.358	0
CDF	0	+ 0.5	-0.393	-0.393		-0.393	+ 0.5	0
PEM	- 7.50	- 7.50	+77.35			-77.35	+ 7.50	+ 7.50
	-19.51	-25.01	-44.84	-44.84		+17.62		
			-13.18		+33.53	+33.53	+13.70	+ 9.35
	+ 2.36	+ 4.72	+ 8.46	+ 8.46		- 3.32		
			- 0.54		+ 2.13	+ 2.13	+ 1.19	+ 0.60
	+ 0.15	+ 0.30	+ 0.54	+ 0.54		- 0.21		
			- 0.05		+ 0.15	+ 0.15	+ 0.03	+ 0.04
	+ 0.01	+ 0.02	+ 0.03	+ 0.03				
$\sum P$	-17.49	-27.47	+27.47			-27.47	+27.47	+17.49
Cal.								
Mom.				-35.31	+35.79			

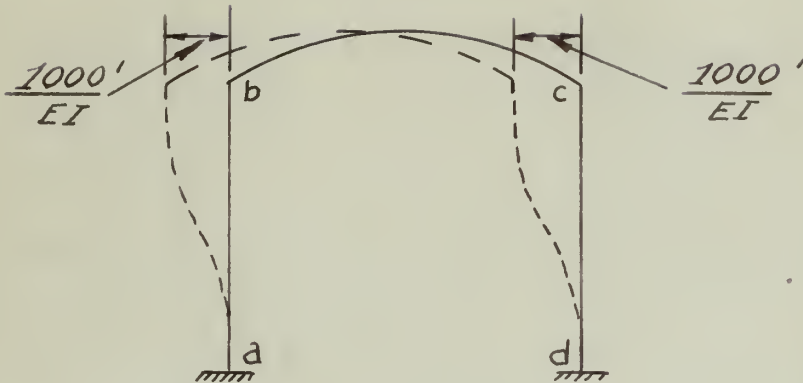
The CJF_{sp} required at B and C = \sum PEM's

PEM Caused by:

	CJF_L	CJF_R
(1) Spread in the arch	(-)28.10	(+)28.10
(2) Balancing moment at B (.21622)(35.31)	(+) 7.74	(-) 7.74
(3) Balancing moment at C (.21622)(35.79)	(+) 7.74	(-) 7.74
(4) Shear in Member AB $\frac{(17.49+27.47)}{20} = \frac{44.96}{20}$	(-) 2.25	
(5) Shear in Member BC		(+) 2.25
$CJF_{sp} =$	(-)14.37	(+)14.37

$$\frac{AJ_{sp}}{CJ_{sp}} (M_{sp} \text{ for } \frac{1000}{EI}) = \frac{7.22}{14.87} = .4855 M_{sp}$$

SIDEWAY



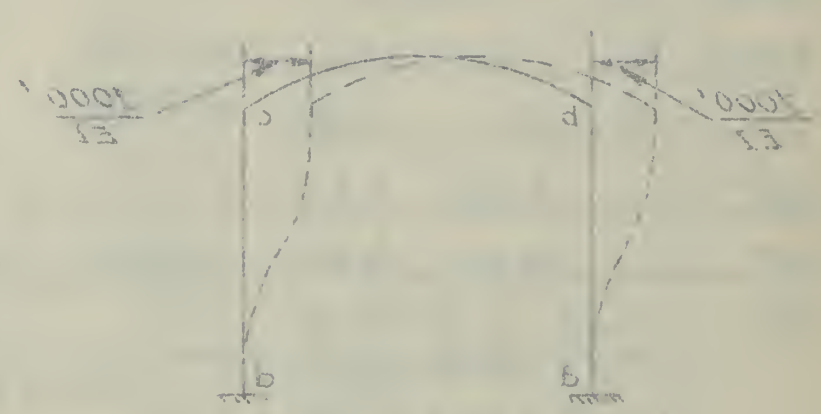
The moments induced in the column members as a result of the assumed sideways equals

$$\frac{6EI}{L^2} = \frac{6EI}{(20)^2} \times \frac{1000}{EI} = 15 \text{ f.k.}$$

$$M_{ab}^f = (-)15 \text{ fk}, M_{ba}^f = (-)15 \text{ fk}, M_{cd}^f = (-)15 \text{ fk}, M_{dc}^f = (-)15 \text{ fk}$$

$$\Delta_{11} = \frac{1}{EI} \int_0^L x^2 dx = \frac{1}{EI} \left[\frac{x^3}{3} \right]_0^L = \frac{L^3}{3EI}$$

2.4.1.1



The structure is subjected to a horizontal load 'P' at the top of each column.

The horizontal displacement of the top of each column is denoted by 'Δ'.

$$\Delta = \frac{PL^2}{2EI}$$

$$\Delta(1) = \frac{PL^2}{2EI} \quad \Delta(2) = \frac{PL^2}{2EI}$$

Solving for the moments at the joints caused by the assumed sideways.

Joint	B				C		
Member	AB	BA	BC	BALANCING	CB	CD	DC
K	0.2	0.2	0.358	MOMENTS	0.358	0.2	0.2
$\frac{v}{L}$	0	0.358	0.642		0.642	0.358	0
CJP	0	+ 0.5	-0.393	-0.393	-0.393	+ 0.5	0
FI	-15.00	-15.00				-15.00	-15.00
	+ 2.69	+ 5.37	+ 9.63	+ 9.63	- 3.72		
			- 4.74		+12.06	+12.06	+ 6.72
	+ 0.85	+ 1.70	+ 3.04	+ 3.04	- 1.18		
			- 0.50		+ 0.76	+ 0.76	+ 0.43
	+ 0.06	+ 0.11	+ 0.19	+ 0.19	- 0.07		
					+ 0.04	+ 0.04	+ 0.03
M_r	-11.40	- 7.82	+ 7.52		+ 7.52	- 7.52	-11.40
Bal.							
Mom.				+12.86	+12.86		

The CJP at B and C = \sum HRF's

HRF Caused by:

		HRF _L \rightarrow HRF _R
(1) Balancing Moment at B (.21622)(12.86)	(-) 2.78	(+) 2.78
(2) Balancing Moment at C (.21622)(12.86)	(+) 2.78	(-) 2.78
(3) Shear in AB $\frac{11.40 + 7.82}{20}$	(-) 0.96	
(4) Shear in BC		(-) 0.96
CJP _{ss}	(-) 0.96 ^k	(-) 0.96 ^k

1. The first step is to find the area of the rectangle.

Area = length \times width

Area = $10 \times 5 = 50$

Area	Length	Width
50	10	5
100	20	5
150	30	5
200	40	5
250	50	5
300	60	5
350	70	5
400	80	5
450	90	5
500	100	5
550	110	5
600	120	5
650	130	5
700	140	5
750	150	5
800	160	5
850	170	5
900	180	5
950	190	5
1000	200	5

Area = length \times width

Area = $10 \times 5 = 50$

Area	Length	Width
50	10	5
100	20	5
150	30	5
200	40	5
250	50	5
300	60	5
350	70	5
400	80	5
450	90	5
500	100	5
550	110	5
600	120	5
650	130	5
700	140	5
750	150	5
800	160	5
850	170	5
900	180	5
950	190	5
1000	200	5

$$\frac{A J R_{ss}}{C J F_{ss}} (M_{ss} \text{ for } \frac{1000}{EI}) = \frac{1.30}{.96} M_{ss} = 1.354 M_{ss}$$

TABULATION OF RESULTS:

Factor	M _{ab}	M _{ba}	M _{bc}	M _{cb}	M _{cd}	M _{oc}
M _r	(+) 2.54	(+) 5.06	(-) 5.06	(-) 2.99	(+) 2.99	(+) 1.49
.4855 M _{sp}	(-) 8.48	(-) 13.34	(+) 13.34	(-) 13.34	(+) 13.34	(+) 8.48
1.354 M _{ss}	(-) 15.44	(-) 10.59	(+) 10.59	(+) 10.59	(-) 10.59	(-) 15.44
Final Internal Moments	(-) 21.38	(-) 18.87	(+) 18.87	(-) 5.74	(+) 5.74	(-) 5.47

$$H_a = \frac{21.38 + 18.87}{20} = \frac{40.25}{20} = 2.013 \text{ k} \rightarrow$$

$$H_d = \frac{(+5.74 - 5.47)}{20} = \frac{(+).27}{20} = .013 \text{ k} \leftarrow$$

Cutting the Bent at "A"

$$V_a(20) + 21.38 - 60 - 40 = (-)5.47$$

$$V_a = \frac{(-)21.38 + 60 + 40 - 5.47}{20} = \frac{73.15}{20} = 3.6575 \text{ k} \uparrow$$

Cutting the Bent at "B"

$$-V_d(20) + 5.47 - 40 + 140 = (-)21.38$$

$$V_d = \frac{(-)21.38 - 140 + 40 - 5.47}{(-)20} = 6.3425 \text{ k} \uparrow$$

$$f_{\text{max}} = \frac{1}{2\pi} \frac{d\phi}{dt} = \frac{1}{2\pi} \frac{d}{dt} \left(\frac{2\pi}{T} t \right) = \frac{1}{T}$$

Frequency of oscillation

Time	Position	Velocity	Acceleration	Force	Displacement
0	0	0	0	0	0
$\frac{T}{4}$	$\frac{A}{2}$	$\frac{A\omega}{2}$	$\frac{A\omega^2}{2}$	$\frac{A\omega^2}{2}$	$\frac{A}{2}$
$\frac{T}{2}$	0	0	0	0	0
$\frac{3T}{4}$	$-\frac{A}{2}$	$-\frac{A\omega}{2}$	$-\frac{A\omega^2}{2}$	$-\frac{A\omega^2}{2}$	$-\frac{A}{2}$
T	0	0	0	0	0

$$v_{\text{max}} = \frac{d}{dt} \left(\frac{A}{2} \sin \omega t \right) = \frac{A\omega}{2} \cos \omega t$$

$$a_{\text{max}} = \frac{d}{dt} \left(\frac{A\omega}{2} \cos \omega t \right) = \frac{A\omega^2}{2} \sin \omega t$$

Frequency of oscillation

$$f = \frac{1}{T} = \frac{1}{2\pi} \frac{d\phi}{dt} = \frac{1}{2\pi} \frac{d}{dt} \left(\frac{2\pi}{T} t \right) = \frac{1}{T}$$

Frequency of oscillation

$$f = \frac{1}{T} = \frac{1}{2\pi} \frac{d\phi}{dt} = \frac{1}{2\pi} \frac{d}{dt} \left(\frac{2\pi}{T} t \right) = \frac{1}{T}$$

APPENDIX A

Prismatic Arch

Sample calculations for determining:

- (1) Reactions at Neutral Point.
- (2) Fixed End Moment.
- (3) Carry-Over Factor.
- (4) Horizontal Thrust Due to Rotation of the Joint.
- (5) Absolute Stiffness.
- (6) Effect of Spread.

APPENDIX

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Introduction and Acknowledgments

1. The Problem of the Problem

2. The Problem of the Problem

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5. The Problem of the Problem

6. The Problem of the Problem

DETERMINATION OF δ_v , δ_{ah} , and α_{ah}
AT NEUTRAL POINT FOR PRISMATIC ARCH

From Formulas Previously Derived

$$\delta_{vv} = \frac{R}{2} [LR - 3(R-r)] \quad \delta_{ah} = \frac{R^2 r^2}{L} - \frac{R}{2} [LR + 3(R-r)] \quad \alpha_{ah} = L$$

Rise/Span = .20 R = .725' R/2 = .3625' L = 1.103473'

$$\frac{LR - 3(R-r)}{.27524775} \quad \frac{LR + 3(R-r)}{1.32524775} \quad \frac{R^2 r^2}{L} \quad \delta_{vv} \quad \delta_{ah} \quad \alpha_{ah}$$

$$.47555439 \quad (+).09977731 \quad (-).00410554 \quad (+)1.103479$$

DETERMINATION OF V_o , H_o and θ at NEUTRAL POINT FOR
PRISMATIC ARCH

rise/span = .20

$\Delta = 43^\circ 36' 11.5'' = .75102$ Radians $\cos \Delta = .72415$ $\sin \Delta = .68965$

Load Point	$\sin 2\theta_{lp}$	θ_{lp}	$\cos 2\theta_{lp}$	θ_{lp} Radians
1	0	0	1.00000	0
2	.15793	7055'41.4"	.99044	.13337
3	.27586	16000'47.2"	.96120	.27948
4	.41379	24026'35.6"	.91037	.42362
5	.55172	33023' 7.2"	.83403	.58443

From Formula Previously Derived

$$V = \frac{R}{4} [LR - 3(R-r)] - \frac{R^3}{8} [\theta_{lp} - \sin \theta_{lp} \cos \theta_{lp}] + R(\theta_{lp})(R-r) - R^2(\theta_{lp}) \cos \theta_{lp}$$

Load Point	θ_{lp}	$R/4 [LR - 3(R-r)]$	$\frac{R^3}{8} (\theta_{lp} - \sin \theta_{lp} \cos \theta_{lp})$	$R\theta_{lp}(R-r)$	$R^2 \theta_{lp} (\cos \theta_{lp})$	ΔV
1	0	.04933365	0	0	0	.04933365
2	.1	.04933365	.00035535	.03802250	.05206000	.03554530
3	.2	.04933365	.00272052	.07612500	.10104615	.02273893
4	.3	.04933365	.00951171	.11413750	.14355397	.01101047
5	.4	.04933365	.02368019	.15225000	.17555431	.00310365

γ δ

$$T = \frac{1}{2} \left[(x-y)^2 + (x+y)^2 \right] = \frac{1}{2} (x^2 - y^2 + x^2 + y^2) = x^2$$

$x = 1, y = 1 \Rightarrow T = 1$

x	y	T
1	1	1
1	0	1
0	1	1
0	0	0

$T = 1$

$x = 1, y = 1 \Rightarrow T = 1$

x	y	T
1	1	1
1	0	1
0	1	1
0	0	0

$T = 1$

$$T = \frac{1}{2} \left[(x-y)^2 + (x+y)^2 \right] = \frac{1}{2} (x^2 - y^2 + x^2 + y^2) = x^2$$

x	y	T
1	1	1
1	0	1
0	1	1
0	0	0

DETERMINATION OF V_o , H_o and ϕ AT NORMAL POINT
FOR PRISMATIC ARCH (Cont.)

$$V_o = \frac{\Delta V}{\delta_{VV}} \quad \delta_{VV} = .00977731$$

Load Point	ΔV	V_o
1	.04988865	(-).5000000
2	.03555580	(-).3555515
3	.02223898	(-).2223860
4	.01101047	(-).1103504
5	.00310385	(-).0311058

From Formula Previously Derived

$$\Delta H = \frac{RS^2}{8} + \frac{R(X_{1p})^2}{8} + R(R-r)\frac{(RS)}{L} - R^2 \cos \phi_{1p} \frac{(RS)}{L} - (R)(X_{1p})\phi_{1p} \frac{(RS)}{L}$$

Load Point	X_{1p}	$\frac{RS}{8}$	$\frac{R(X_{1p})^2}{8}$	$\frac{R(X_{1p})^2}{8}$	$R(R-r)\frac{RS}{L}$	$R^2 \cos \phi_{1p} \frac{(RS)}{L}$
1	0	.65701295	.090625	0	.25007555	.54534243
2	.1	.65701295	.090625	.003625	.25007555	.54204094
3	.2	.65701295	.090625	.014500	.25007555	.53194311
4	.3	.65701295	.090625	.032625	.25007555	.51438937
5	.4	.65701295	.090625	.058000	.25007555	.48802594

X_{1p}	$R(X_{1p})(\phi_{1p})\frac{RS}{L}$	ΔH	δ_{hh}	H_o
0	0	(-).00464188	(-).00410584	(-).1.1305555
.1	.00659104	(-).00430543	(-).00410584	(-).1.0436539
.2	.02682519	(-).00336775	(-).00410584	(-).8202531
.3	.06096413	(-).00202795	(-).00410584	(-).4939184
.4	.11135364	(-).00057903	(-).00410584	(-).1653515

From Formula Previously Derived

$$\phi = R(R-r) + R^2(\cos \phi_{1p}) - \frac{(X_{1p})^2}{8} + R(X_{1p})\phi_{1p}$$

$$H_o = \frac{\phi}{\alpha_{HH}}$$

n	$P_n(x)$	$P_n'(x)$	$P_n''(x)$
0	1	0	0
1	x	1	0
2	$x^2 - 1$	$2x$	2
3	$x^3 - 3x$	$3x^2 - 3$	$6x$

TABLE 2. THE FIRST FOUR COEFFICIENTS OF THE POLYNOMIALS $Q_n(x)$ FOR $n = 0, 1, 2, 3$

n	$Q_n(x)$	$Q_n'(x)$	$Q_n''(x)$	$Q_n'''(x)$	$Q_n^{(4)}(x)$	$Q_n^{(5)}(x)$
0	1	0	0	0	0	0
1	x	1	0	0	0	0
2	$x^2 - 1$	$2x$	2	0	0	0
3	$x^3 - 3x$	$3x^2 - 3$	$6x$	6	0	0

n	$P_n(x)$	$Q_n(x)$	$R_n(x)$	$S_n(x)$	$T_n(x)$
0	1	1	1	1	1
1	x	x	x	x	x
2	$x^2 - 1$	$x^2 - 1$	$x^2 - 1$	$x^2 - 1$	$x^2 - 1$
3	$x^3 - 3x$	$x^3 - 3x$	$x^3 - 3x$	$x^3 - 3x$	$x^3 - 3x$

TABLE 3. THE FIRST FOUR COEFFICIENTS OF THE POLYNOMIALS $U_n(x)$ FOR $n = 0, 1, 2, 3$

n	$U_n(x)$	$U_n'(x)$	$U_n''(x)$	$U_n'''(x)$
0	1	0	0	0
1	x	1	0	0
2	$x^2 - 1$	$2x$	2	0
3	$x^3 - 3x$	$3x^2 - 3$	$6x$	6

DETERMINATION OF V_o , H_o and ϕ AT NEUTRAL POINT
FOR PRISMATIC ARCH (Cont.)

Load Point	x_{1p}	$R(R-r)$	$R^2 \cos^2 \alpha_{1p}$	$\frac{K_{1p} L}{2}$	$R(x_{1p})(\alpha_{1p})$
1	0	.5206250	.5256250	0	0
2	.1	.3806250	.5206000	.08517395	.01003183
3	.2	.3806250	.5082307	.11034790	.04052460
4	.3	.3806250	.4785132	.16552185	.09278985
5	.4	.3806250	.4383370	.22099580	.16948470

Load Point	ϕ	H_o
1	.1450000	.1314026
2	.0943328	.0859398
3	.0547324	.0498452
4	.0251582	.0227972
5	.0065508	.0059356

DETERMINATION OF FIXED END MOMENTS FOR PARABOLIC ARCH

$$M_1^f = V_o \frac{3}{2} - H_o [r - (R - \bar{y})] + M_o$$

$$M_r^f = V_o \frac{3}{2} - H_o [r - (R - \bar{y})] + M_o + 1 \left(\frac{3}{2} - x_{1p} \right)$$

x_{1p}	$V_o \frac{3}{2}$	$H_o [r - (R - \bar{y})]$	M_o
0	(-.5000000)(-.5)	(-1.1305555)(-.1320130)	(-.1514026)
.1	(-.3563515)(-.5)	(-1.0488539)(-.1320130)	(-.0659393)
.2	(-.2228860)(-.5)	(-0.9202331)(-.1320130)	(-.0496452)
.3	(-.1103904)(-.5)	(-0.4939194)(-.1320130)	(-.0287972)
.4	(-.0311058)(-.5)	(-0.1653815)(-.1320130)	(-.0059355)

M_1^f	M_r^f
(-).030651	(-).030651
(-).046224	(-).002575
(-).046454	(+).030630
(-).032826	(+).056824
(-).012216	(+).056673

The Fixed End Moments shown above are caused by 1 kip loads placed at distances x_{1p} to the right of the crown. F.E.M. (left) as caused by a 1^k load to the right of the crown equals F.E.M. (right) as caused by a 1^k load placed at the corresponding point to the left of the crown. All F.E.M. values were tabulated on this basis.

$$I^2 = \frac{1}{2} \left[(I_1^2 + I_2^2) - (I_1 - I_2)^2 \right]$$

$$I^2 = \frac{1}{2} \left[(I_1^2 + I_2^2) - (I_1 - I_2)^2 \right]$$

I	I^2	I^2	I^2
(1000000, -)	(1000000, -) (1000000, -)	(1000000, -)	1
(1000000, -)	(1000000, -) (1000000, -)	(1000000, -)	1
(1000000, -)	(1000000, -) (1000000, -)	(1000000, -)	1
(1000000, -)	(1000000, -) (1000000, -)	(1000000, -)	1
(1000000, -)	(1000000, -) (1000000, -)	(1000000, -)	1
(1000000, -)	(1000000, -) (1000000, -)	(1000000, -)	1

I	I^2
(1000000, -)	(1000000, -)
(1000000, -)	(1000000, -)
(1000000, -)	(1000000, -)
(1000000, -)	(1000000, -)
(1000000, -)	(1000000, -)

The first two columns show the values of I and I^2 for the first two cases. The third column shows the values of I and I^2 for the third case. The fourth column shows the values of I and I^2 for the fourth case. The fifth column shows the values of I and I^2 for the fifth case. The sixth column shows the values of I and I^2 for the sixth case. The seventh column shows the values of I and I^2 for the seventh case. The eighth column shows the values of I and I^2 for the eighth case. The ninth column shows the values of I and I^2 for the ninth case. The tenth column shows the values of I and I^2 for the tenth case.

DETERMINATION OF CARRY OVER FACTOR,
THRUST, AND ABSOLUTE SLIPPERNESS (ITERATIVE)

Span Length = 1.0 Rise/Span = .20 R = .725' \widehat{L} = 1.103479

Carry Over

$$(1) EI \delta'_{vm} = EI \alpha'_{mv} = \widehat{L} S/2 = .58173950$$

$$(2) EI \delta'_{hm} = EI \alpha'_{mh} = \widehat{L} [Rise - (R-r)] \\ = 1.103479 [.2 - (.725 - .85701295)] = .14567352$$

$$(3) EI \alpha_{mm} = \widehat{L} = 1.1034790$$

$$(4) EI \delta_{vv} = \frac{S^2 \widehat{L}}{4} + S/2 [\widehat{LR} - S(R-r)] \\ = \frac{(1.103479)^2}{4} + \frac{.725}{2} [.725(1.103479) - 1(.725 - .2)] = .37564705$$

$$(5) EI \delta_{hh} = S/2 [\widehat{LR} + S(R-r)] + (R-r)^2 \widehat{L} - 2R(R-r) S \\ = .43040233 + .30414640 - .761250 = + .02329873$$

$$(6) EI \delta_{vh} = EI \delta_{hv} = \widehat{L} S/2 [Rise - (R-r)] = S/2 (2) = .07253676$$

$$D_2 = \frac{(EI)^2 [(\delta'_{vm})(\delta_{hh}) - (\delta'_{hm})(\delta_{vv})]}{(EI)^2 [(\delta_{hv})(\delta_{vh}) - (\delta_{vv})(\delta_{hh})]} = \frac{(1)(5) - (2)(6)}{(6)(6) - (4)(5)}$$

$$D_2 = \frac{.01985463 - .01061039}{.00530519 - .00375210} = -.6511470$$

$$V_a = D_2 W_a = -.6511470 W_a$$

$$-P_o = -(1+D_2)(3)W_a = -(1-.651147)W_a = -.348853W_a$$

$$\text{Carry Over Factor} = -.348853$$

THEORY OF THE EARTH'S CRUST

By J. H. VAN DER KAMPT, D. Sc., Professor of Geology, University of Amsterdam

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DETERMINATION OF CARRY OVER FACTOR,
THRUST AND ABSOLUTE STIFFNESS (PRISMATIC)

Thrust

$$D_1 = \frac{(EI)^2 [(\delta'_{vm})(\delta_{hv}) - (\delta'_{hm})(\delta_{vv})]}{(EI)^2 [(\delta_{hh})(\delta_{vv}) - (\delta_{hv})(\delta_{vh})]} = \frac{(1)(6) - (2)(4)}{(5)(4) - (0)(6)}$$

$$= \frac{.04018692 - .05472183}{.00875210 - .00530513} = (-)4.216799$$

$$M_a = D_1 M_a = -4.216799 M_a$$

Absolute Stiffness

From Formulas Previously Derived

$$\text{When } \theta_a = 1 \quad M_a = \frac{EI}{D_3} \quad \text{where } D_3 = (\alpha_{vm} + D_2 \alpha'_{ev} + D_1 \alpha'_{eh}) EI$$

$$(EI)(\alpha_{vm}) = \widehat{L} = 1.103479$$

$$(EI)(D_2)(\alpha'_{ev}) = (D_2)(\frac{\widehat{L}}{2}) = (-.651147)(.5517395) = -.35926352$$

$$(EI)(D_1)(\alpha'_{eh}) = (D_1)[\widehat{L}(\text{rise} - [h - \bar{y}])] = (-4.216799)(.14557352) =$$

$$= -.61427595$$

$$D_3 = 1.103479 - .35926352 - .61427595 = +.1299395$$

$$\text{Absolute Stiffness} = M_a = \frac{EI}{D_3} = 7.695889 EI$$

PROBLEM 10.10
Find the Laplace transform of the function

$$f(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 1 & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \int_0^1 e^{-st} t dt + \int_1^2 e^{-st} dt$$

$$= \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right]_0^1 + \left[-\frac{1}{s} e^{-st} \right]_1^2$$

$$= \left(-\frac{1}{s} e^{-s} - \frac{1}{s^2} e^{-s} \right) - \left(-\frac{1}{s^2} \right) + \left(-\frac{1}{s} e^{-2s} + \frac{1}{s} e^{-s} \right)$$

$$= -\frac{1}{s} e^{-s} - \frac{1}{s^2} e^{-s} + \frac{1}{s^2} - \frac{1}{s} e^{-2s} + \frac{1}{s} e^{-s}$$

$$= \frac{1}{s^2} - \frac{1}{s} e^{-s} - \frac{1}{s^2} e^{-s} + \frac{1}{s} e^{-s} - \frac{1}{s} e^{-2s}$$

$$= \frac{1}{s^2} - \frac{1}{s} e^{-2s}$$

$$\therefore \mathcal{L}\{f(t)\} = \frac{1}{s^2} - \frac{1}{s} e^{-2s}$$

$$\text{Hence, the Laplace transform of } f(t) \text{ is } \frac{1}{s^2} - \frac{1}{s} e^{-2s}$$

$$\text{Ans. } \mathcal{L}\{f(t)\} = \frac{1}{s^2} - \frac{1}{s} e^{-2s}$$

DETERMINATION OF MOMENT AND REACTIONS
DUE TO THE EFFECT OF STIFFNESS (PRISMATIC)

Moment

From Formulas Previously Derived

$$M_a = \frac{EI \Delta H}{D_4} \quad \text{where } D_4 = EI \delta'_{hm} - \frac{(\delta'_{vm})(\delta_{hh})(EI)^2}{(\delta_{vh})(EI)}$$

$$(1) \quad EI \delta'_{hm} = .14567352$$

$$(2) \quad (EI \delta'_{vm})(EI \delta_{hh}) = (.55173950)(.02329873) = .01255483$$

$$(3) \quad EI \delta_{vh} = .07253676$$

$$(4) \quad D_4 = (1) - \frac{(2)}{(3)} = .14567352 - \frac{.01255483}{.07253676} = -.0303147$$

$$(5) \quad M_a = \frac{EI \Delta H}{(-).0303147} = -32.45204 EI \Delta H$$

Vertical Reaction

From Formulas Previously Derived

$$V_a = 0$$

Horizontal Reaction

From Formulas Previously Derived

$$H_a = \frac{EI \Delta H}{D_5} \quad \text{where } D_5 = EI \delta_{hh} - \frac{(\delta_{vh})(\delta'_{hm})(EI)^2}{EI \delta'_{vm}}$$

$$(1) \quad EI \delta_{hh} = .02329873$$

$$(2) \quad (EI \delta_{vh})(EI \delta'_{hm}) = (.07253676)(.14567352) = .01061039$$

$$(3) \quad EI \delta'_{vm} = .55173950$$

$$(4) \quad D_5 = (1) - \frac{(2)}{(3)} = +.00406794$$

$$(5) \quad H_a = \frac{EI \Delta H}{.00406794} = (+)245.8247 EI \Delta H$$

PROBLEM 1. THE LIMIT OF THE SEQUENCE
(PROBLEM 1. THE LIMIT OF THE SEQUENCE)

PROBLEM

THE LIMIT OF THE SEQUENCE

$$\frac{n(1 + \frac{1}{n})^n - 1}{1 + \frac{1}{n}} = \frac{1}{n} \left(1 + \frac{1}{n} \right)^n - \frac{1}{n} \quad (1)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(1 + \frac{1}{n} \right)^n = 0 \quad (2)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(1 + \frac{1}{n} \right)^n = 0 \quad (3)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(1 + \frac{1}{n} \right)^n = 0 \quad (4)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(1 + \frac{1}{n} \right)^n = 0 \quad (5)$$

$$\Delta \left(\frac{1}{n} \left(1 + \frac{1}{n} \right)^n \right) = \frac{1}{n} \left(1 + \frac{1}{n} \right)^n - \frac{1}{n+1} \left(1 + \frac{1}{n+1} \right)^{n+1} \quad (6)$$

PROBLEM 2. THE LIMIT OF THE SEQUENCE

THE LIMIT OF THE SEQUENCE

$$x = \frac{1}{n}$$

PROBLEM 3. THE LIMIT OF THE SEQUENCE

THE LIMIT OF THE SEQUENCE

$$\frac{n(1 + \frac{1}{n})^n - 1}{1 + \frac{1}{n}} = \frac{1}{n} \left(1 + \frac{1}{n} \right)^n - \frac{1}{n} \quad (1)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(1 + \frac{1}{n} \right)^n = 0 \quad (2)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(1 + \frac{1}{n} \right)^n = 0 \quad (3)$$

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$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(1 + \frac{1}{n} \right)^n = 0 \quad (5)$$

$$\Delta \left(\frac{1}{n} \left(1 + \frac{1}{n} \right)^n \right) = \frac{1}{n} \left(1 + \frac{1}{n} \right)^n - \frac{1}{n+1} \left(1 + \frac{1}{n+1} \right)^{n+1} \quad (6)$$

APPENDIX B

NON-PRISMATIC ANCH

Sample Calculations for Determining:

- (1) Reactions at Neutral Point.
- (2) Fixed End Moment.
- (3) Carry-Over Factor.
- (4) Horizontal Thrust Due to Rotation of
the Joint.
- (5) Absolute Stiffness.
- (6) Effect of Spread.

EXHIBIT 6

EXHIBIT 6

Exhibit 6 - [illegible]

(1) [illegible]

(2) [illegible]

(3) [illegible]

(4) [illegible]

(5) [illegible]

(6) [illegible]

(7) [illegible]

DETERMINATION OF δ_{vv} , δ_{hh} and α_{nn} AT NEUTRAL POINT FOR NON-PRISMATIC ARCH

From Formulas Previously Derived

$$\delta_{vv} = \frac{s^3}{12} \quad \delta_{vv} = \left[\left(\frac{y}{s} \right)^2 - \frac{r^2}{s^2} \right] s + \frac{s^3}{12} \quad \alpha_{nn} = s$$

$$\text{Rise/Span} = .20 \quad R = .725' \quad R/s = .3625' \quad \hat{L} = 1.103479'$$

$$\delta_{vv} = .003333 \quad \delta_{hh} = .0032812 \quad \alpha_{nn} = 1.00000$$

DETERMINATION OF V_o , H_o and θ AT NEUTRAL POINT FOR NON-PRISMATIC ARCH

$$\text{Rise/Span} = .20$$

From Formula Previously Derived

$$V_o = -\frac{\Delta V}{\delta_{vv}} = -\left[.5 + 2\left(\frac{x_{lp}}{s}\right)^3 - 3/2\left(\frac{x_{lp}}{s}\right) \right]$$

Load Point	x_{lp}	$2\left(\frac{x_{lp}}{s}\right)^3$	$3/2\left(\frac{x_{lp}}{s}\right)$	V_o
1	0	-	-	.500
2	.1	.002	.15	.352
3	.2	.016	.30	.216
4	.3	.054	.45	.104
5	.4	.128	.60	.028

From Formula Previously Derived

$$\Delta H = (-)\left(\frac{R-r}{s}\right)^3 - \frac{y}{s}\left(\frac{s^2}{s} + \frac{x_{lp}^2}{2}\right) + \frac{R^3}{3}\cos^3\theta_{lp} + \frac{R^2 x_{lp}}{2}(\theta_{lp} + \sin\theta_{lp}\cos\theta_{lp})$$

For values $\sin\theta_{lp}$, θ_{lp} , $\cos\theta_{lp}$ See Page

THEORY OF THE FUNCTION OF THE COMPLEX VARIABLE

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{where } a_n = \frac{f^{(n)}(0)}{n!}$$

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$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{where } a_n = \frac{f^{(n)}(0)}{n!}$$

z	$f(z)$	$f'(z)$	$f''(z)$	$f'''(z)$
0	1	0	0	0
1	2	1	0	0
2	4	2	0	0
3	8	3	0	0
4	16	4	0	0

THEORY OF THE FUNCTION OF THE COMPLEX VARIABLE

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{where } a_n = \frac{f^{(n)}(0)}{n!}$$

THEORY OF THE FUNCTION OF THE COMPLEX VARIABLE

DETERMINATION OF V_o , H_o and e AT NEUTRAL POINT
FOR NON-PRISMATIC ARCH (Cont.)

x_{1p}	$\bar{y}(\frac{s^2}{8} + \frac{x_{1p}^2}{2})$	$\frac{r^3}{3} \cos^3 \phi_{1p}$	$\frac{r^2}{2} x_{1p}$	$\phi_{1p} + \sin \phi_{1p} \cos \phi_{1p}$	$\frac{(R-r)^3}{3}$
0	.08202799	.12708604	0	0	.04823437
.1	.08614110	.12341765	.02628125	.27428	.04823437
.2	.09608046	.11280648	.05256250	.54464	.04823437
.3	.11264606	.09583995	.07884375	.80332	.04823437
.4	.13583790	.07369496	.10512500	1.04458	.04823437

Load Point	ΔH	δ_{hh}	H_o
1	.00403632	.0032212	1.2530465
2	.00373100	.0032212	1.1582640
3	.00288071	.0032212	.8942971
4	.00170372	.0032212	.5229064
5	.00056584	.0032212	.1756612

From Formula Previously Derived

$$e = \frac{s^2}{8} + \frac{(x_{1p})^2}{2} - \frac{x_{1p}s}{2}$$

$$H_o = -\frac{e}{\alpha_{em}}$$

Load Point	x_{1p}	$\frac{s^2}{8}$	$\frac{(x_{1p})^2}{2}$	$\frac{x_{1p}s}{2}$	e	H_o
1	0	.125	-	0	.125	.125
2	.1	.125	.005	.05	.020	.080
3	.2	.125	.020	.10	.045	.045
4	.3	.125	.045	.15	.020	.020
5	.4	.125	.080	.20	.005	.005

DETERMINATION OF FIXED END MOMENT FOR NON-PRISMATIC ARCH

$$M_1^f = V_o \frac{S}{2} - H_o [r - (R - \gamma)] + M_o$$

$$M_2^f = V_o \frac{S}{2} - H_o [r - (R - \gamma)] + M_o + 1 \left(\frac{S}{2} - X_{1p} \right)$$

X_{1p}	$V_o \frac{S}{2}$	$H_o [r - (R - \gamma)]$	M_o
0	(-.5000000)(-.5)	(-1.2530485)(-.1376239)	-.1250000
.1	(-.3620000)(-.5)	(-1.1582640)(-.1376239)	-.0600000
.2	(-.2160000)(-.5)	(-.8942971)(-.1376239)	-.0450000
.3	(-.1040000)(-.5)	(-.5289084)(-.1376239)	-.0200000
.4	(-.0280000)(-.5)	(-.1756612)(-.1376239)	-.0050000

M_1^f	M_2^f
-.047443	-.047443
-.063405	-.016405
-.060077	+.023283
-.040790	+.055210
-.015175	+.086825

The Fixed End Moments shown above are caused by 1 kip loads placed at distances X_{1p} to the right of the crown. F.E.M. (left) as caused by a 1^k load to the right of the crown equals F.E.M. (right) as caused by a 1^k load placed at the corresponding point to the left of the crown. All F.E.M. values were tabulated on this basis.

DETERMINATION OF CARRY OVER FACTOR, THRUST,
AND ABSOLUTE STIFFNESS (NON-PRISMATIC)

$$\text{Span Length} = 1.0 \quad \text{rise/span} = .20 \quad r = .725' \quad \widehat{L} = 1.103479$$

Carry Over

$$(1) EI_c \delta'_{vm} = EI_c \alpha'_{mv} = \frac{r^2}{2} = .5000$$

$$(2) EI_c \delta'_{hm} = EI_c \alpha'_{mh} = \frac{1}{2} [\widehat{L}r - 2(R-r)] = \frac{1}{2} [(1.103479)(.725) - (1)(.525)] \\ = .13762588$$

$$(3) EI_c \alpha'_{mh} = 3 = 1.000$$

$$(4) EI_c \delta'_{vv} = \frac{r^3}{3} = .33333$$

$$(5) EI_c \delta'_{hh} = 3R^2 - \frac{r^3}{12} - \widehat{L}r(R-r) = .52562500 - .00333333 - .42013007 \\ = .022161600$$

$$(6) EI_c \delta'_{vh} = EI_c \delta'_{hv} = \frac{r}{4} [\widehat{L}r - 3(R-r)] = \frac{1}{4} [(1.103479)(.725) - (1)(.525)] \\ = .06881124$$

$$D_2' = \frac{(EI_c)^2 [(\delta'_{vm})(\delta'_{hh}) - (\delta'_{hm})(\delta'_{vh})]}{(EI_c)^2 [(\delta'_{hv})(\delta'_{vh}) - (\delta'_{vv})(\delta'_{hh})]} = \frac{(1)(5) - (2)(6)}{(6)(5) - (4)(5)}$$

$$D_2' = \frac{.01109080 - .00947017}{.00473508 - .00733720} = .6073012$$

$$V_a = D_2' M_a = -.6073012 M_a$$

$$-M_b = -(1+D_2')(S)M_a = -(1-.6073012)(1)M_a = -.3926988 M_a$$

Thrust

$$D_1' = \frac{(EI_c)^2 [(\delta'_{vm})(\delta'_{hv}) - (\delta'_{hm})(\delta'_{vv})]}{(EI_c)^2 [(\delta'_{bh})(\delta'_{vv}) - (\delta'_{hv})(\delta'_{vh})]} = \frac{(1)(6) - (2)(4)}{(5)(4) - (6)(5)} \\ = \frac{.03440597 - .04587462}{.00733720 - .00473508} = -4.324337$$

$$H_a = D_1' M_a = -4.324337 M_a$$

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1967-1968

$$[1] \quad \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right)$$

$$[2] \quad \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right)$$

1967-1968

$$[3] \quad \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right)$$

$$[4] \quad \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right)$$

$$[5] \quad \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right)$$

1967-1968

$$[6] \quad \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right)$$

1967-1968

$$\frac{(1/2) - (1/2)}{(1/2) - (1/2)} = \frac{[1/2] - [1/2]}{[1/2] - [1/2]} = 1$$

$$[7] \quad \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right)$$

$$[8] \quad \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right)$$

$$[9] \quad \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right)$$

1967-1968

$$\frac{(1/2) - (1/2)}{(1/2) - (1/2)} = \frac{[1/2] - [1/2]}{[1/2] - [1/2]} = 1$$

$$[10] \quad \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right)$$

$$[11] \quad \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{2} \right)$$

DETERMINATION OF CARRY OVER FACTOR,
THRUST, AND ABSOLUTE STIFFNESS (BOS-PRIMEAIC)

Absolute Stiffness

From Formulas Previously Derived

$$\text{When } \alpha_4 = 1 \quad K_a = \frac{EI_c}{D_3} \quad \text{where } D_3 = (\alpha_{2m} + D_2' \alpha_{2v}' + D_1' \alpha_{2h}') EI_c$$

$$(EI_c)(\alpha_{2m}) = 3 = 1.000$$

$$(EI_c)(D_2')(\alpha_{2v}') = (-.60730120)(.5) = -.30365060$$

$$(EI_c)(D_2')(\alpha_{2h}') = (-4.324337)(.13762383) = -.59513204$$

$$D_3' = 1.000 - .30365060 - .59513204 = +.10121736$$

$$\text{Absolute Stiffness} = K_a = \frac{EI_c}{D_3'} = 9.8797281 EI_c$$

THEORY OF THE EARTH'S CRUST
 (PART I) - THE CRUSTAL THICKNESS

THE CRUSTAL THICKNESS

The crustal thickness is a function of the crustal density and the crustal thickness.

$$h = \frac{g}{\rho_c - \rho_m} \quad (1)$$

$$h = \frac{g}{\rho_c - \rho_m} \quad (2)$$

$$h = \frac{g}{\rho_c - \rho_m} \quad (3)$$

$$h = \frac{g}{\rho_c - \rho_m} \quad (4)$$

$$h = \frac{g}{\rho_c - \rho_m} \quad (5)$$

$$h = \frac{g}{\rho_c - \rho_m} \quad (6)$$

DETERMINATION OF MOMENT AND REACTIONS
DUE TO THE EFFECT OF SPREAD (NON-PRISMATIC)

Moment

From Formulas Previously Derived

$$M_a = \frac{EI_c \Delta H}{D_4'} \quad \text{where } D_4' = \frac{(EI_c \delta_{vh})(EI_c \delta'_{hm}) - (\delta'_{vm})(\delta_{hh})(EI_c)^2}{(EI_c)(\delta_{vh})}$$

$$(1) (EI_c \delta'_{hm})(EI_c \delta_{vh}) = (.13762388)(.06881194) = .00947017$$

$$(2) (EI_c \delta'_{vm})(EI_c \delta_{hh}) = (.5)(.02216160) = .01108080$$

$$(3) (EI_c \delta_{vh}) = .06881194$$

$$(4) D_4' = \frac{(1)-(2)}{(3)} = \frac{.00947017 - .01108080}{.06881194} = -.02340631$$

$$(5) M_a = \frac{EI_c \Delta H}{D_4'} = \frac{EI_c \Delta H}{-.02340631} = -42.723521 EI_c \Delta H$$

Vertical Reaction

From Formulas Previously Derived

$$V_a = 0$$

Horizontal Reaction

From Formulas Previously Derived

$$H_a = \frac{EI_c \Delta H}{D_5'} \quad \text{where } D_5' = \frac{(EI_c \delta'_{vm})(EI_c \delta_{hh}) - (EI_c \delta'_{hm})(EI_c \delta_{vh})}{EI_c \delta'_{vm}}$$

$$(1) (EI_c \delta'_{vm})(EI_c \delta_{hh}) = (.5)(.02216160) = .01108080$$

$$(2) (EI_c \delta'_{hm})(EI_c \delta_{vh}) = (.13762388)(.06881194) = .00947017$$

$$(3) EI_c \delta'_{vm} = .5$$

$$(4) D_5' = \frac{(1)-(2)}{(3)} = \frac{.01108080 - .00947017}{.5} = +.00322127$$

$$(5) H_a = \frac{EI_c \Delta H}{D_5'} = \frac{EI_c \Delta H}{.00322127} = (+)310.43575 EI_c \Delta H$$

General

from the following provisions:

$$\frac{(\delta_{11} - \delta_{12})}{\delta_{11}} = \frac{\Delta_{11}}{\delta_{11}} = \delta$$

$$\delta_{11} = (\delta_{11} - \delta_{12}) + \delta_{12} = \delta_{12} + \delta_{11} \quad (1)$$

$$\delta_{12} = (\delta_{11} - \delta_{12}) + \delta_{12} = \delta_{12} + \delta_{11} \quad (2)$$

$$\delta_{11} = \delta_{11} + \delta_{12} \quad (3)$$

$$\delta_{11} = \frac{\delta_{11} - \delta_{12}}{\delta_{11}} = \frac{\delta_{11} - \delta_{12}}{\delta_{11}} = \delta \quad (4)$$

$$\Delta_{11} = \frac{\delta_{11} - \delta_{12}}{\delta_{11}} = \frac{\delta_{11} - \delta_{12}}{\delta_{11}} = \delta \quad (5)$$

Special

from the following provisions:

$$\delta = \delta$$

General

from the following provisions:

$$\frac{(\delta_{11} - \delta_{12})}{\delta_{11}} = \frac{\Delta_{11}}{\delta_{11}} = \delta$$

$$\delta_{11} = (\delta_{11} - \delta_{12}) + \delta_{12} = \delta_{12} + \delta_{11} \quad (1)$$

$$\delta_{12} = (\delta_{11} - \delta_{12}) + \delta_{12} = \delta_{12} + \delta_{11} \quad (2)$$

$$\delta_{11} = \delta_{11} + \delta_{12} \quad (3)$$

$$\delta_{11} = \frac{\delta_{11} - \delta_{12}}{\delta_{11}} = \frac{\delta_{11} - \delta_{12}}{\delta_{11}} = \delta \quad (4)$$

$$\Delta_{11} = \frac{\delta_{11} - \delta_{12}}{\delta_{11}} = \frac{\delta_{11} - \delta_{12}}{\delta_{11}} = \delta \quad (5)$$

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MEMORANDUM

TO : THE SECRETARY OF DEFENSE - ROOM 3000 - 1400

FROM : THE SECRETARY OF DEFENSE - ROOM 3000 - 1400

SUBJECT: MEMORANDUM FOR THE SECRETARY OF DEFENSE

DATE: 10/10/50

RE: MEMORANDUM FOR THE SECRETARY OF DEFENSE

DATE: 10/10/50

TO : THE SECRETARY OF DEFENSE - ROOM 3000 - 1400

FROM : THE SECRETARY OF DEFENSE - ROOM 3000 - 1400

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Thesis
B24
c.1

Barron

Application of moment distribution to prismatic and non-prismatic circular arched bents.

12846

Thesis
B24
c.1

Barron

Application of moment distribution to prismatic and non-prismatic circular arched bents.

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